Patterns with Bounded Treewidth

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Pattern languages

$x_1 \; \mathtt{aa} \; x_2 \; x_1 \; x_2 \; \mathtt{cb} \; x_1$

Pattern languages

x_1 aa x_2 x_1 x_2 cb x_1

acaaabcbaacabcbacbac

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$\{w \mid w = u \, \mathrm{aa} \, v \, u \, v \, \mathrm{cb} \, u, \text{ where } u, v \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}^*\}.$

The membership problem for pattern languages

Theorem

The membership problem for pattern languages is NP-complete (Angluin, 80).

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Main research task: Find classes of pattern languages with a polynomial membership problem.

Previous results

- Brute-force algorithm with runtime $O(|w|^k)$, where k is the number of variables.
- A pattern language is a regular language if
  - it is unary or
  - no variable in the pattern is repeated.
- $|w| \leq k$ for a constant k (Geilke, Zilles, ALT 2011)
- *Non-cross* patterns (e. g., $x_1 x_1 x_1 x_2 x_2 x_3 x_3 x_3$) (Shinohara, 82).
- Patterns with bounded *variable distance* (R., S., CIAA 2010).

Relational Structures

Definition (Relational structure)

A relational vocabulary τ is a finite set of relation symbols. Every relation symbol $R \in \tau$ has an arity $\mathrm{ar}(R) \geq 1$. A τ-structure $\mathcal{A}$ comprises a finite set A called the universe and, for every $R \in \tau$, an interpretation $R^{\mathcal{A}} \subseteq A^{ar(R)}$.

Example

Every graph can be interpreted as a relational structure $\mathcal{G}$, where the universe V is the set of vertices and the edges are given as an interpretation of a binary relation E.

We only consider relations with arity of at most 2.

Homomorphism problem for relational structures

Definition (Homomorphism)

Let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures with universes A and B, respectively. A *homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ is a mapping $h : A \to B$ such that for all $R \in \tau$ and for all $a_1, a_2, \ldots, a_{\mathrm{ar}(R)} \in A$, $(a_1, a_2, \ldots, a_{\mathrm{ar}(R)}) \in R^{\mathcal{A}}$ implies $(h(a_1), h(a_2), \ldots, h(a_{\mathrm{ar}(R)})) \in R^{\mathcal{B}}$.

Definition (Homomorphism problem)

The *homomorphism problem HOM* is the problem of deciding, for any structures $\mathcal{A}$ and $\mathcal{B}$, whether there exists a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. For any set of structures C, by $\mathrm{HOM}(C)$ we denote the homomorphism problem, where the left hand structure is restricted to be from C.

Homomorphism problem for relational structures

Theorem

The homomorphism problem for relational structures is NP-complete.

α-structures

$$\alpha := x_1 \quad x_2 \quad x_1 \quad x_3 \quad x_2 \quad x_3 \quad \mathtt{b} \quad x_1 \quad x_2 \quad x_1$$

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1 2 3 4 5 6 7 8 9 10

α-structures

$$\alpha := x_1 \quad x_2 \quad x_1 \quad x_3 \quad x_2 \quad x_3 \quad \mathtt{b} \quad x_1 \quad x_2 \quad x_1$$

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 10$$

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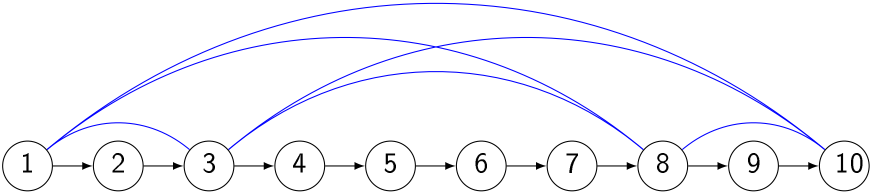

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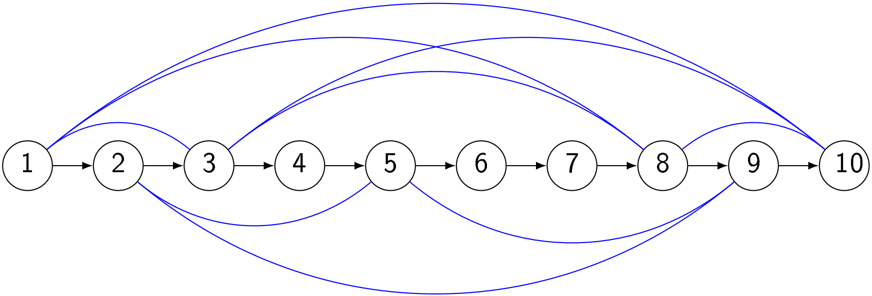

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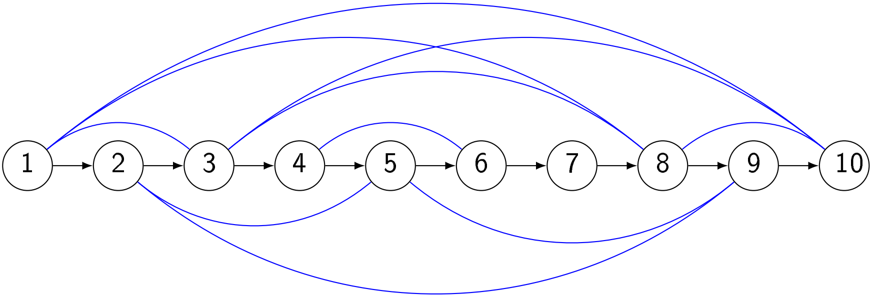

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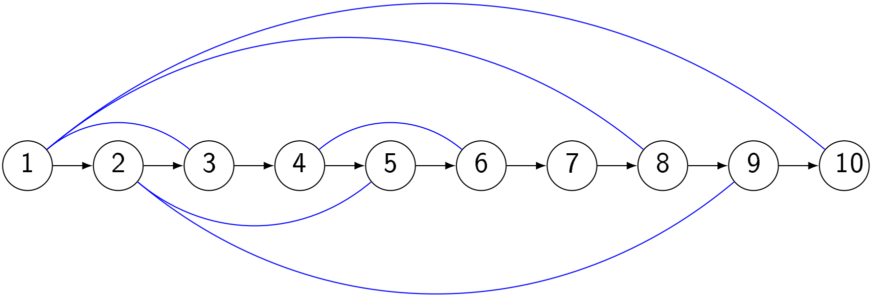

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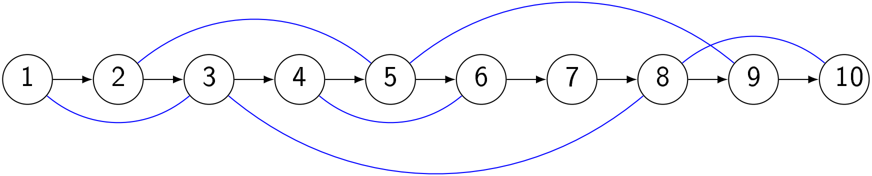

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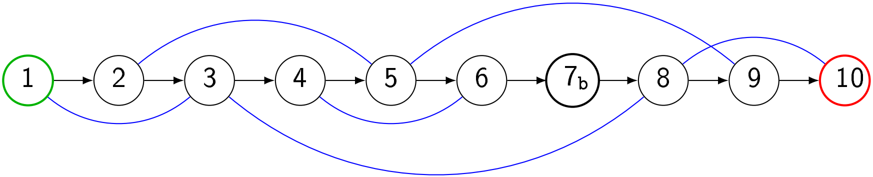

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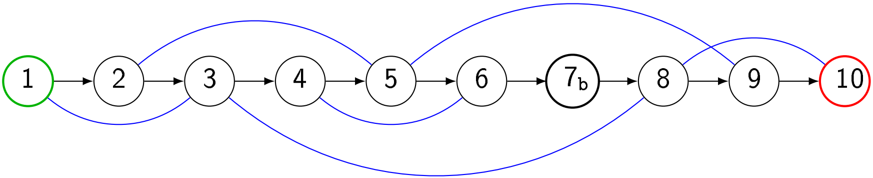

We denote α-structures by $\mathcal{A}_\alpha$ and the standard α-structure by $\mathcal{A}^s_\alpha$.

w-structures

$w := \texttt{abab}$

w-structures

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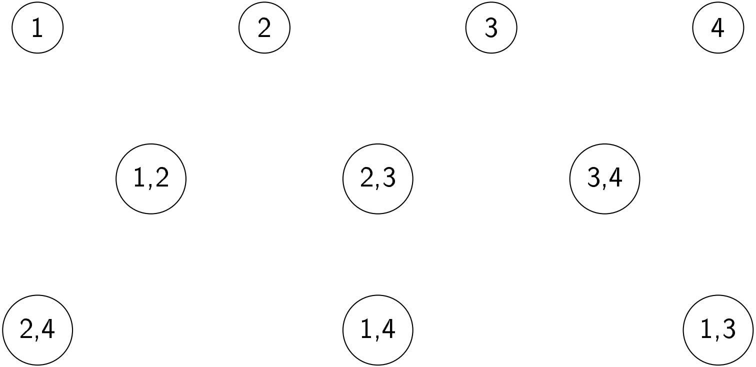

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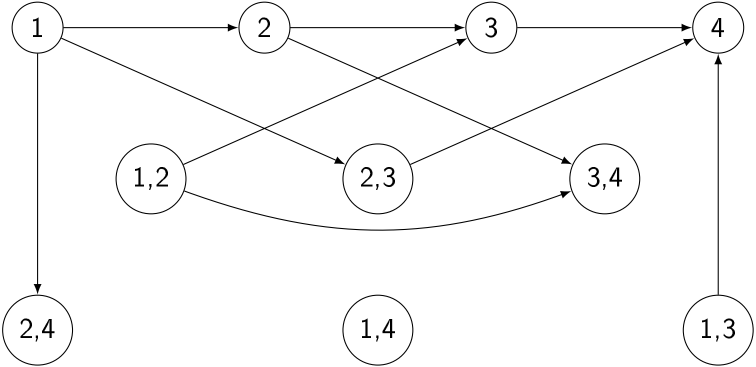

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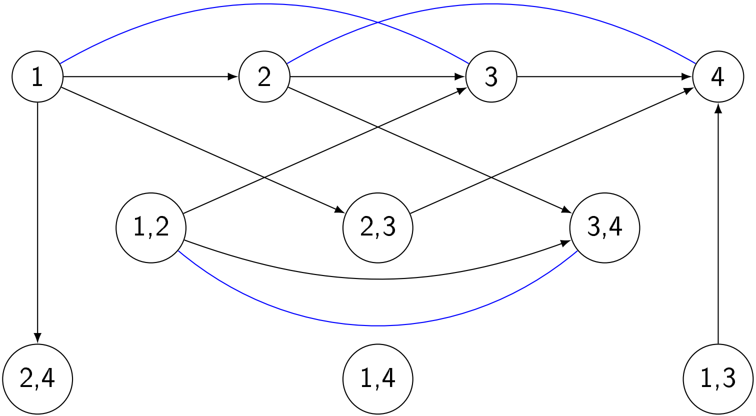

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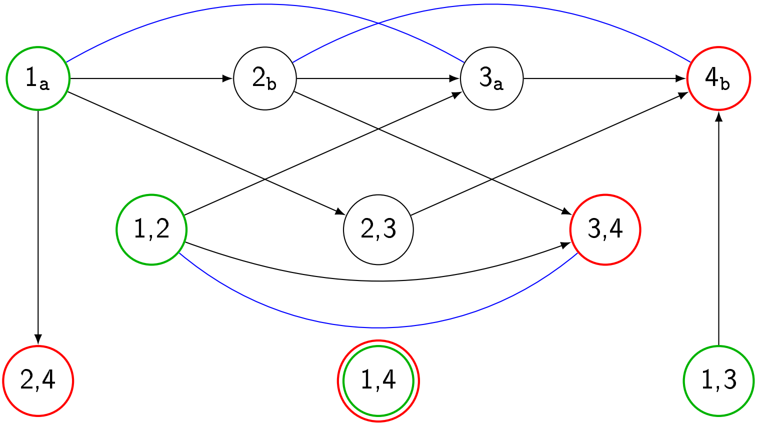

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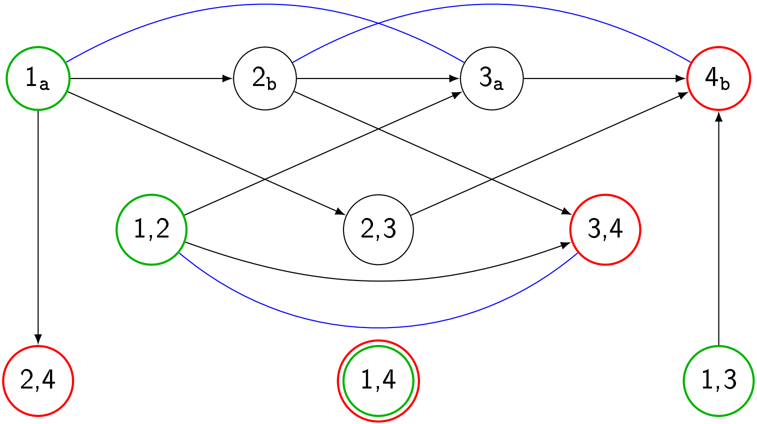

We denote w-structures by $\mathcal{A}_w$.

Membership problem $\leq$ homomorphism problem

Lemma

A word w is in the pattern language of α if and only if there exists a homomorphism from $\mathcal{A}_\alpha$ to $\mathcal{A}_w$, where $\mathcal{A}_\alpha$ is some α-structure.

A meta theorem

Theorem (Freuder, 90)

Let C be a class of relational structures with bounded treewidth. Then HOM(C) can be solved in polynomial time.

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Theorem (Meta theorem)

Let P be a class of patterns and let f be a polynomial time computable function that maps every $\alpha \in P$ to an α-structure. If, for some constant k, $\max\{\mathrm{tw}(f(\alpha)) \mid \alpha \in P\} \leq k$, then the membership problem for P is decidable in polynomial time.

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Research task: Find classes of patterns with bounded treewidth.

Scope coincidence degree

Let α be a pattern.

- For every $y \in \mathsf{var}(\alpha)$, the *scope of y in α* is defined by $\mathsf{sc}_\alpha(y) := \{i, i+1, \ldots, j\}$, where i is the leftmost and j the rightmost position of y in α.
- The scopes of $y_1, y_2, \ldots, y_k \in \mathsf{var}(\alpha)$ *coincide in α* if and only if $\bigcap_{1 \leq i \leq k} \mathsf{sc}_\alpha(y_i) \neq \emptyset$.
- The *scope coincidence degree* of α ($\mathsf{scd}(\alpha)$) is the maximum number of variables in α such that their scopes coincide.

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$$\alpha_1 := x_1 \quad x_2 \quad x_1 \quad x_3 \quad x_2 \quad x_3 \quad x_1 \quad x_2 \quad x_3$$

$$\alpha_2 := x_1 \quad x_2 \quad x_1 \quad x_1 \quad x_2 \quad x_3 \quad x_2 \quad x_3 \quad x_3$$

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- The *scope coincidence degree* of α ($\text{scd}(\alpha)$) is the maximum number of variables in α such that their scopes coincide.

$$\alpha_1 := x_1 \; x_2 \; x_1 \; x_3 \; x_2 \; x_3 \; x_1 \; x_2 \; x_3 \qquad \text{scd}(\alpha_1) = 3$$

$$\alpha_2 := x_1 \; x_2 \; x_1 \; x_1 \; x_2 \; x_3 \; x_2 \; x_3 \; x_3 \qquad \text{scd}(\alpha_2) = 2$$

Scope coincidence degree

Lemma

Let α be a pattern. Then $\mathrm{tw}(\mathcal{A}^s_\alpha) \leq \mathrm{scd}(\alpha) + 1$.

Theorem

Let $k \in \mathbb{N}$ and $P := \{\alpha \mid \mathrm{scd}(\alpha) \leq k\}$. The membership problem for the class P is decidable in polynomial time.

Mildly entwined patterns

Let α be a pattern.

- Two variables $x, y \in \text{var}(\alpha)$ are *entwined* iff $\alpha = \beta \cdot x \cdot \gamma_1 \cdot y \cdot \gamma_2 \cdot x \cdot \gamma_3 \cdot y \cdot \delta$.
- If no two variables are entwined, then α is *nested*.
- α is *closely entwined* iff $\alpha = \beta \cdot x \cdot \gamma_1 \cdot y \cdot \gamma_2 \cdot x \cdot \gamma_3 \cdot y \cdot \delta$ with $|\gamma_2|_x = |\gamma_2|_y = 0$ implies $\gamma_2 = \varepsilon$.
- α is *mildly entwined* iff it is closely entwined and, for every $x \in \text{var}(\alpha)$, if $\alpha = \beta \cdot x \cdot \gamma \cdot x \cdot \delta$ with $|\gamma|_x = 0$, then γ is nested.

Mildly entwined patterns

$$\alpha = \quad x_1 \; x_3 \; x_4 \; x_4 \; x_3 \; x_3 \; x_1 \; x_2 \; x_3 \; x_5 \; x_5 \; x_2 \; x_5 \; x_6 \; x_6 \; x_2$$

Mildly entwined patterns

$$\alpha = \quad x_1 \quad x_3 \quad x_4 \quad x_4 \quad x_3 \quad x_3 \quad x_1 \quad x_2 \quad x_3 \quad x_5 \quad x_5 \quad x_2 \quad x_5 \quad x_6 \quad x_6 \quad x_2$$

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- for every $x \in \mathrm{var}(\alpha)$, if $\alpha = \beta \cdot x \cdot \gamma \cdot x \cdot \delta$ with $|\gamma|_x = 0$, then γ is nested.

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$$\alpha = \quad x_1 \; x_3 \; x_4 \; x_4 \; x_3 \; x_3 \; x_1 \; x_2 \; x_3 \; x_5 \; x_5 \; x_2 \; x_5 \; x_6 \; x_6 \; x_2$$

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$$\alpha = \quad x_1 \quad x_3 \quad x_4 \quad x_4 \quad x_3 \quad x_3 \quad x_1 \quad \color{red}{x_2} \quad x_3 \quad x_5 \quad x_5 \quad \color{red}{x_2} \quad x_5 \quad x_6 \quad x_6 \quad x_2$$

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- for every $x \in \mathrm{var}(\alpha)$, if $\alpha = \beta \cdot x \cdot \gamma \cdot x \cdot \delta$ with $|\gamma|_x = 0$, then γ is nested.

$\Rightarrow \alpha$ is mildly entwined.

Mildly entwined patterns

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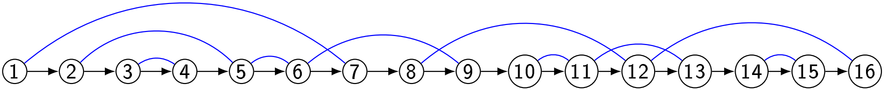

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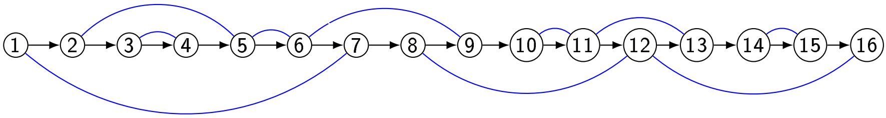

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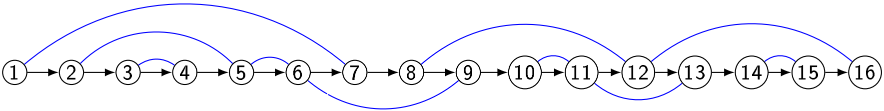

Mildly entwined patterns

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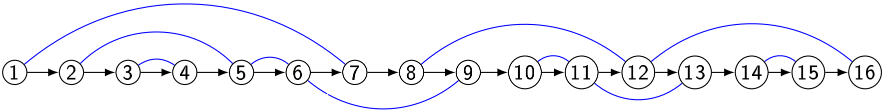

Definition

A graph is *outerplanar* iff it can be drawn in a planar way such that no vertex is entirely surrounded by edges (or, equivalently, all vertices lie on the exterior face).

Mildly entwined patterns

Lemma

A pattern α is mildly entwined if and only if $\mathcal{A}^s_\alpha$ is outerplanar.

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Theorem (Bodlaender 86)

If $\mathcal{G}$ is an outerplanar graph, then $\mathrm{tw}(\mathcal{G}) \leq 2$.

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If $\mathcal{G}$ is an outerplanar graph, then $\mathrm{tw}(\mathcal{G}) \leq 2$.

Theorem

Let $P := \{\alpha \mid \alpha$ is mildly entwined$\}$. The membership problem for the class P is decidable in polynomial time.

Mildly entwined vs. bounded scope coincidence degree

Observation

For every k, $k \geq 2$, $\{\alpha \mid \mathrm{scd}(\alpha) \leq k\} \# \{\alpha \mid \alpha \text{ is mildly entwined}\}$.