

On the Parameterized Complexity of Default Logic and Autoepistemic Logic

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- models common-sense reasoning
- extends classical logic with *default rules*

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- models common-sense reasoning
- extends classical logic with *default rules*
- undecidable for first order logic (Reiter)
- **here:** propositional logic

Default Rules and Theories

Definition (Reiter 80)

A **default rule** is a triple $\frac{\alpha : \beta}{\gamma}$, where

- α is called the **prerequisite**,
- β is called the **justification**, and
- γ is called the **consequent**,

for α, β, γ propositional formulae.

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Informally: infer a formula γ from a set of formulae W by a default rule

$\frac{\alpha : \beta}{\gamma}$, if

- α is derivable from W and
- β is consistent with W .

Default Theories

Definition (Reiter 80)

A **default theory** is a tuple $\langle W, D \rangle$, where W is a set of formulae and D is a set of default rules.

Example: Playing Football with Default Rules

$$W := \{ \text{football}, \text{rain}, \text{cold} \wedge \text{rain} \rightarrow \text{snow} \}$$

$$D := \left\{ \frac{\text{football} : \neg \text{snow}}{\text{takesPlace}} \right\}$$

$\neg \text{snow}$ is consistent with W . Hence we can infer *takesPlace*.

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$\neg \text{snow}$ is inconsistent with W . Hence we **cannot** infer *takesPlace*.

Default logics are **non-monotone**!

Semantics: Stable Extensions

Definition (Reiter 80)

For default theory $\langle W, D \rangle$ and set of formulae E , we define $\Gamma(E)$ as the smallest set, s.t.

- 1 $W \subseteq \Gamma(E)$,
- 2 $\Gamma(E)$ is closed under deduction, and
- 3 for all defaults $\frac{\alpha : \beta}{\gamma}$ with $\alpha \in \Gamma(E)$ and $\neg\beta \notin E$, it holds that $\gamma \in \Gamma(E)$.

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Stable extensions correspond to possible views of an agent on the basis of $\langle W, D \rangle$.

Generating Defaults

Semantics: Generating Defaults (Reiter 80)

Given: a default theory $\langle W, D \rangle$ and set of formulae E :

Define the set of **generating defaults** as

$$G := \left\{ \frac{\alpha : \beta}{\gamma} \in D \mid \alpha \in E \text{ and } \neg\beta \notin E \right\}.$$

Then: E is stable a extension of $\langle W, D \rangle$ **iff**

$$E = \text{Th} \left(W \cup \left\{ \gamma \mid \frac{\alpha : \beta}{\gamma} \in G \right\} \right).$$

The Decision Problem

Extension Existence Problem

Instance: a default theory $\langle W, D \rangle$

Question: Does $\langle W, D \rangle$ have a stable extension?

Theorem (Gottlob 92)

The Extension Existence Problem is Σ_2^P -complete.

Boolean function, B -formula

n -ary Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$

Let B be a set of Boolean functions.

A B -formula is a formula whose connectives are functions from B

$\mathcal{L}(B)$ denotes the set of all B -formulae

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$$B = \{\vee, \neg, f, g\}$$

$$\left. \begin{array}{l} \neg x \\ x \vee \neg y \\ \neg(x \vee y) \vee z \\ \neg\neg\neg x \\ f(g(x, y), f(x, g(z, y), y), z) \end{array} \right\} \in \mathcal{L}(B)$$

Parameterized Complexity Theory

Definition (FPT)

- $Q \subseteq \Sigma^*$ a decision problem
- $\kappa : \Sigma^* \rightarrow \mathbb{N}$ associates a parameter to the instances

\mathbb{A} is an **fpt-algorithm** for (Q, κ) if there is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$, a polynomial p s.t. \mathbb{A} decides Q and for every input x , \mathbb{A} 's runtime is at most $f(k) \cdot p(|x|)$, where $k = \kappa(x)$.

(Q, κ) is **fixed-parameter tractable** if there is an fpt-algorithm that decides (Q, κ) . All such problems are in the class FPT.

tree width

- a parameter definable on any relational structure
- graph: intuitively it measures the distance of the graph from a tree
- many NP-hard problems become FPT with this parameter
- this parameter is not computable in polynomial time (but FPT)

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Powerfull Theorem by Courcelle: all MSO-definable problems become FPT with the tree width parameter

A handy tool

Definition

Let Q be a decision problem, x be an instance.

Q is **MSO-definable** iff there exists an MSO-formula ϕ_Q and a function $x \mapsto \mathcal{A}_x$ s.t. it holds that $x \in Q$ if and only if $\mathcal{A}_x \models \phi_Q$.

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Theorem (Courcelle 1990, Elberfeld Jakoby Tantau 2010)

Let Q be an MSO-definable decision problem, and let \mathcal{A}_x be the structure associated with an instance x . Further let $k \in \mathbb{N}$ s.t. the treewidth of \mathcal{A}_x is bounded by k . Then Q is solvable in time $O(f(k) \cdot |x|)$ and space $O(\log(f(k)) + \log |x|)$.

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Describe a problem with MSO-formula ϕ .

Apply Courcelle's Theorem and get an FPT-result.

Express a Set of Boolean Functions Γ in MSO

- Let B be a set of Boolean functions. Define the Vocabulary as

$$\begin{aligned}\tau_B &:= \{\text{const}_f^1 \mid f \in B, \text{arity}(f) = 0\} \cup \\ &\quad \{\text{conn}_{f,i}^2 \mid f \in B, 1 \leq i \leq \text{arity}(f)\}, \\ \tau_{B,prop} &:= \tau_B \cup \{\text{variable}^1, \text{formula}^1\}.\end{aligned}$$

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With Γ a set of B -formulae we associate the $\tau_{B,prop}$ -structure \mathcal{A}_Γ .

Universe: Formulae and their subformulae.

Expressing Satisfiability

Lemma

Let B be a finite set of Boolean functions. Then there exists an MSO-formula θ_{sat} over $\tau_{B,prop}$ such that for $\Gamma \subseteq \mathcal{L}(B)$

Γ is satisfiable iff $\mathcal{A}_\Gamma \models \theta_{sat}$.

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Proof (sketch)

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Let $f(1) = 0$, $g(0, 1) = 1$. $\theta_{assign}(M)$ demands that, if $x \in M$, then $g(f(x), x) \in M$ and $f(x) \notin M$.

Expressing Implication

Let B be a set of Boolean functions and F, G be sets of B -formulae. Extend our vocabulary to express $F \models G$:

$$\tau_{B,imp} := \tau_{B,prop} \cup \{\text{formula}_{pre}^1, \text{formula}_{conc}^1\},$$

where $\text{formula}_{pre}^1(x)$ is true iff x represents a formula from F , and $\text{formula}_{conc}^1(x)$ is true iff x represents a formula from G .

Lemma

Let B be a finite set of Boolean functions. Then there exists an MSO-formula θ_{imp} over $\tau_{B,imp}$ such that for any $\Gamma \subseteq \mathcal{L}(B)$ and any $\Delta_1, \Delta_2 \subseteq \Gamma$ it holds that there exists a structure \mathcal{A}_Γ such that

$$\Delta_1 \models \Delta_2 \text{ iff and } \mathcal{A}_\Gamma \models \theta_{imp}.$$

Proof.

Define the MSO-formulae $\theta_{premise}(M)$, $\theta_{conclusion}(M)$, and $\theta_{implies}$ as follows:



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Then, we can define the formulae θ_{imp} as $\theta_{imp} := \theta_{struc} \wedge \theta_{implies}$. □


On the Way to Default Logic: Extend the Vocabulary

Default Theory (W, D) , where $D := \left\{ \frac{\alpha_i : \beta_i}{\gamma_i} \mid 1 \leq i \leq n \right\}$.

$$\tau_{B,dl} := \tau_{B,imp} \cup \{kb^1, def^1, prem^2, concl^2, just^2\}$$

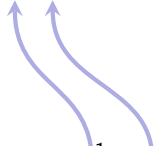
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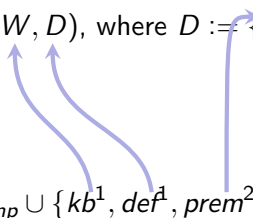
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The diagram consists of five blue arrows pointing upwards from the set definition to the default theory definition. The first arrow points from kb^1 to the W in (W, D) . The second arrow points from def^1 to the D in (W, D) . The third arrow points from pre^2 to the α_i in the numerator of the fraction. The fourth arrow points from $concl^2$ to the γ_i in the denominator of the fraction. The fifth arrow points from $just^2$ to the β_i in the numerator of the fraction.

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Extension Existence of Default Logic

Lemma

Let B be a finite set of Boolean functions, (W, D) a B -default theory and $\mathcal{A}_{(W,D)}$ the associated structure. Then there exists an MSO-formula $\theta_{extension}$ such that (W, D) possesses a stable extension iff $\mathcal{A}_{(W,D)} \models \theta_{extension}$.

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Proof.

$\theta_{extension} := \theta_{struc} \wedge \exists G(\theta_{stable}(G))$, where

$\theta_{stable}(G) := \forall d(\text{def}(d) \rightarrow (G(d) \leftrightarrow \theta_{app}(d, G)))$ □

Apply Courcelle's Theorem

Theorem

Let B be a finite set of Boolean functions, let $k \in \mathbb{N}$ be fixed, and let (W, D) be a B -default theory such that the treewidth of structures for $\theta_{\text{extension}}$ is bounded by k .

Then the extension existence problem for B -default logic is solvable in time $O(f(k) \cdot |(W, D)|)$ and space $O(\log(f(k)) + \log |(W, D)|)$.

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Lower bounds

Theorem

There exists a family of (very simple) default theories $(\emptyset, D)_k$ such that the tree width of $\mathcal{A}_{(\emptyset, D)_k}$ is not constant.

Transfer to Autoepistemic Logic

- Introduced by Moore 1985.
- New modal operator L to model the beliefs of a perfect rational agent.
- Satisfiability and reasoning Σ_2^P -complete.

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Theorem

Let B be a finite set of Boolean functions, let $k \in \mathbb{N}$ be fixed, and let Σ be a set of autoepistemic B -formulae such that \mathcal{A}_Σ has tree width bounded by k . Then the expansion problem is solvable in time $O(f(k) \cdot |\Sigma|)$ and space $O(\log(f(k)) + \log |\Sigma|)$.

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- Thinking about useful kinds of parameterizations of propositional Default Logic beyond treewidth.
- And the same for autoepistemic logic.

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Thank you!