Computational Complexity of Rule Distributions of Non-Uniform Cellular Automata

Alberto Dennunzio[1] Enrico Formenti[2] Julien Provillard[2]

[1]Università degli Studi di Milano-Bicocca,
Dipartimento di Informatica, Sistemistica e Comunicazione,
Viale Sarca 336, 20126 Milano (Italy).

[2]Université Nice-Sophia Antipolis,
Laboratoire I3S,
2000 Route des Colles, 06903 Sophia Antipolis (France).

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Motivations

- Wide literature. Applications in biology, physics, cryptography, . . .
- General framework for non-uniform cellular automata.
- Distributions of local rules *vs.* automata properties.

Outline

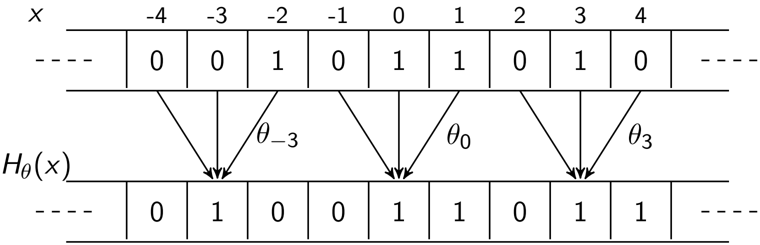
Outline

Non-Uniform Cellular Automata (ν-CA)

Definition

A *non-uniform cellular automaton* is a couple (A, θ) where A is a finite alphabet and θ a distribution of local rules of radius r on A. It defines a global transition function H_θ by

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, H_\theta(x)_i = \theta_i(x_{[i-r,i+r]})$$



Characterization of Properties

- $\mathcal{R}$ = finite set of local rules of radius r
- Θ = distributions on $\mathcal{R}$
- If P is a proposition on non-uniform cellular automata,

$$\mathcal{L}_P = \{\theta \in \Theta : H_\theta \text{ verifies } P\}$$

Outline

Backround

- $A = \{0, 1, \ldots, s-1\}$
- Finite configurations
- Partial charge: $\mu_n(x) = \sum_{i=-n}^{n} x_i$
- Global charge: $\mu(x) = \lim_{n \to \infty} \mu_n(x)$

Number Conservation

Definition

- A ν-CA H is *number conserving on finite configurations* (FNC) if for all finite configuration x, $\mu(x) = \mu(H(x))$.
- A ν-CA H is *number-conserving* (NC) if
  - $H(\underline{0}) = \underline{0}$
  - $\forall x \in A^{\mathbb{Z}} \setminus \{\underline{0}\}, \quad \lim_{n\to\infty} \frac{\mu_n(H(x))}{\mu_n(x)} = 1.$

Proposition

H is NC if and only if it is NFC.

Characterization

Proposition

$P(H) =$ "H is NC". Then $\mathcal{L}_P$ is a SFT.

$\mathcal{L}_P = X_{\mathcal{F}}$

$$\mathcal{F} = \{\psi \in \mathcal{R}^{2r+1} : \exists u \in A^{2r+1}, \psi_{2r}(u) \neq u_0 + \sum_{i=0}^{2r-1} \psi_{i+1}(0^{2r-i} u_{[1,i+1]}) - \psi_i(0^{2r-i} u_{[0,i]})\} \ .$$

Outline

Surjectivity and Finite Distributions

Proposition

H_θ is surjective iff $h_{\theta_{[i,j]}}$ is surjective $\forall i \leq j$.

Corollary

$P(H) =$ "H is surjective". Then $\mathcal{L}_P$ is a subshift.

$$\mathcal{F} = \{\psi \in \mathcal{R}^* : h_\psi \text{ is not surjective}\} \ .$$

De Bruijn Graphs

Definition

The *De Bruijn graph of* $\mathcal{R}$ is graph $\mathcal{G} = (V, E)$, where $V = A^{2r}$ and edges in E are all the pairs (aw, wb) with label $(f, f(awb))$, obtained varying $a, b \in A$, $w \in A^{2r-1}$, and $f \in \mathcal{R}$.

Proposition

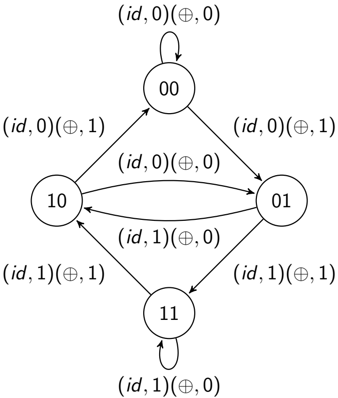
- *A bi-infinite path defines a distribution* θ *and two configurations* x *and* y *such that* $H_\theta(x) = y$*.*
- *A finite path defines a finite distribution* ψ *and two words* v *and* w *such that* $h_\psi(v) = w$*.*

Example

Let *id* and $\oplus$ be the local rules of radius 1 on $A = \{0, 1\}$ such that

- $id(x, y, z) = y$
- $\oplus(x, y, z) = x + z \pmod 2$

The associated De Bruijn Graph is



Characterization of surjectivity

Proposition

$\mathcal{L}_P$ is a sofic subshift.

The language of a De Bruijn graph is

$$\mathcal{L} = \{(\psi, u) \in (\mathcal{R} \times A)^* : h_\psi^{-1}(u) \neq \emptyset\}$$

Then

$$\mathcal{L}^c = \{(\psi, u) \in (\mathcal{R} \times A)^* : h_\psi^{-1}(u) = \emptyset\}$$

and

$$\mathcal{L}' = \{\psi \in \mathcal{R}^* : \exists u \in A^*, h_\psi^{-1}(u) = \emptyset\} = \mathcal{F}$$

are rational.

Injectivity and Product Graph

Definition

Let $\mathcal{G} = (V, E)$ be the De Bruijn graph of $\mathcal{R}$. The *product graph* $\mathcal{P}$ of $\mathcal{R}$ is the labeled graph $(V \times V, W)$ where $((u, u'), (v, v')) \in W$ with label $(f, a) \in \mathcal{R} \times A$ if and only if $(u, v), (u', v') \in E$ both with the same label (f, a).

Proposition

A bi-infinite path defines a distribution θ and three configurations x, y and z such that $H_\theta(x) = H_\theta(y) = z$.

Injectivity

Proposition

$P(H) =$ "H is injective". Then $\mathcal{L}_P$ is a ζ-rationnal language.

$$\{\theta \in \Theta : \exists x \neq y \in A^{\mathbb{Z}}, H_\theta(x) = H_\theta(y)\} = \{\theta \in \Theta : H_\theta \text{ is not injective}\}$$

is a ζ-rationnal language then its complementary $\mathcal{L}_P$ is also ζ-rationnal.

Outline

Definitions

Definition

- H is *sensitive* iff

$$\exists n \geq 0, \forall x \in A^{\mathbb{Z}}, \forall m \geq 0, \exists y \in A^{\mathbb{Z}}, x_{[-m,m]} = y_{[-m,m]} \text{ and } \exists k \geq 0, H^k(x)_{[-n,n]} \neq H^k(y)_{[-n,n]} \ .$$

- $x \in A^{\mathbb{Z}}$ is an *equicontinuity point* of H iff

$$\forall n \geq 0, \exists m \geq 0, \forall y \in A^{\mathbb{Z}}, x_{[-m,m]} = y_{[-m,m]} \Rightarrow \forall k \geq 0, H^k(x)_{[-n,n]} = H^k(y)_{[-n,n]} \ .$$

- H is *equicontinuous* iff all its points are equicontinuity points

Definitions

- $(A, +, .)$ = finite commutative ring
- f linear iff $\forall u \in A^{2r+1}, f(u) = \sum_{i=0}^{2r} \lambda_i.u_i$
- H_θ linear iff θ_i linear $\forall i$

Proposition

H linear iff

$$\forall x, y \in A^{\mathbb{Z}}, \forall a \in A, H(a.x + y) = a.H(x) + H(y)$$

Proposition

Linear ν-CA are either sensitive or equicontinuous

Walls

Definition (Wall)

A *right-wall* is any element $\psi \in \mathcal{R}^*$ of length $n \geq r$ such that, for all word $v \in A^r$,

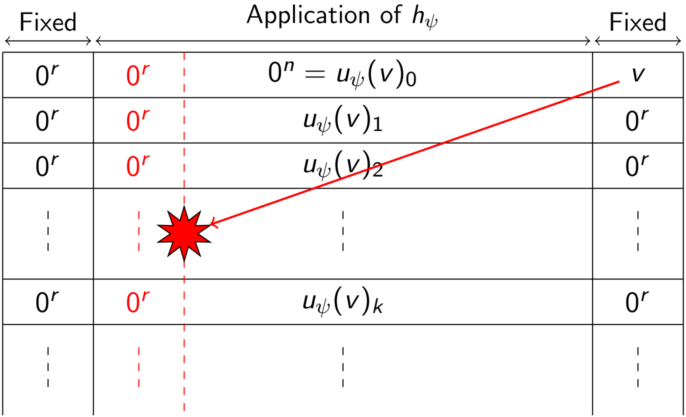
$$\begin{aligned} u_\psi(v)_0 &= 0^n \\ u_\psi(v)_1 &= h_\psi(0^r u_\psi(v)_0 v) \\ u_\psi(v)_{k+1} &= h_\psi(0^r u_\psi(v)_k 0^r) \text{ for } k > 1 \end{aligned}$$

verifies $\forall k \in \mathbb{N}, (u_\psi(v)_k)_{[0,r-1]} = 0^r$. *Left-walls* are defined similarly.

Walls

Fixed	Application of h_ψ	Fixed
0^r	$0^n = u_\psi(v)_0$	v
0^r	$u_\psi(v)_1$	0^r
0^r	$u_\psi(v)_2$	0^r
$\vdots$	$\vdots$	$\vdots$
0^r	$u_\psi(v)_k$	0^r
$\vdots$	$\vdots$	$\vdots$

Walls


Fixed
Application of h_ψ
Fixed
0^r
0^r
$0^n = u_\psi(v)_0$
v
0^r
0^r
$u_\psi(v)_1$
0^r
0^r
0^r
$u_\psi(v)_2$
0^r
0^r
0^r
$u_\psi(v)_k$
0^r

Characterization

Proposition

H_θ sensitive iff one of the two following conditions holds

1. $\exists n \in \mathbb{N}$ *such that for all integer* $m \geq n + r$, $\theta_{[n+1,m]}$ *is not a right-wall.*
2. $\exists n \in \mathbb{N}$ *such that for all integer* $m \leq -n - r$, $\theta_{[m,-n-1]}$ *is not a left-wall.*

Case of the radius 1

$$f(a,b,c) = \lambda_f^-.a + \tilde{\lambda}_f.b + \lambda_f^+.c$$

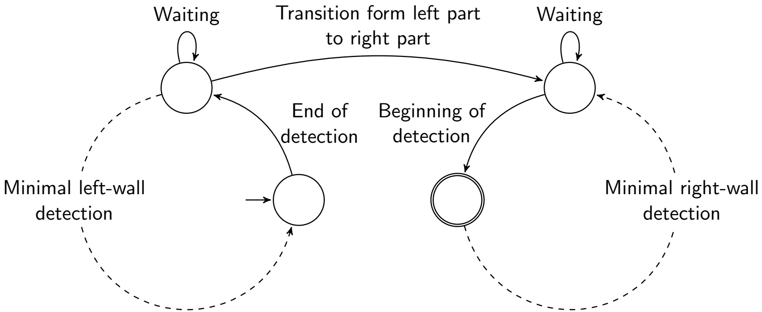
Proposition

1. ψ *is a right-wall if and only if* $\prod_{i=0}^{n-1} \lambda_{\psi_i}^+ = 0$.
2. ψ *is a left-wall if and only if* $\prod_{i=0}^{n-1} \lambda_{\psi_i}^- = 0$.

Case of the radius 1

Proposition

$P(H) =$ "H is equicontinuous". Then $\mathcal{L}_P$ is a ζ-rationnal language.



Conclusions

- Complexity of properties according to the classes of languages they involve.
- Characterizing the equicontinuity for any radius and/or non-commutative rings.
- Exploring more in-depth classes of bi-infinite languages.

Thanks for your attention!