Defining contexts in context-free grammars

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(*) If $w \in \Sigma^*$ is representable as $X_1 \cdot \ldots \cdot X_\ell$, then w has the property A.

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The idea deserves a correct implementation!

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$$A \to \alpha_1 \& \ldots \& \alpha_m \quad (\alpha_1, \ldots, \alpha_m \in (\Sigma \cup N)^*)$$

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Successful extension of inductive definitions beyond context-free grammars.

Grammars with one-sided contexts

$G = (\Sigma, N, R, S)$, with rules of the form

$$A \to \alpha_1 \,\&\, \dots \,\&\, \alpha_k \,\&\, \lhd\beta_1 \,\&\, \dots \,\&\, \lhd\beta_m \,\&\, \trianglelefteq\gamma_1 \,\&\, \dots \,\&\, \trianglelefteq\gamma_n$$

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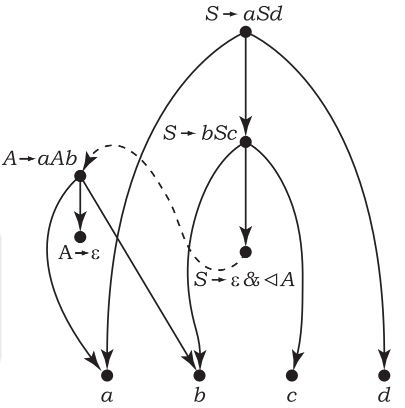

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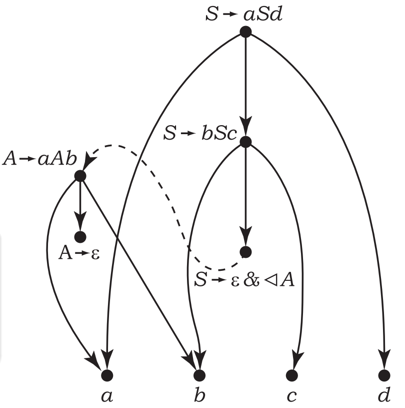

✓ More involved example: "declaration before or after use".

Definition by deduction

Elementary proposition $[\alpha, \langle u \rangle v]$.

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$$[B, \langle x \rangle u], [C, \langle xu \rangle v], [D, \langle \varepsilon \rangle x] \vdash_G [A, \langle x \rangle uv] \qquad (\text{for } x, u, v \in \Sigma^*).$$

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Define

$$L_G(A) = \{ \langle u \rangle v \mid \vdash_G [A, \langle u \rangle v] \},$$
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- A proof tree is a parse tree.

Example revisited

Grammar for $\{ a^n b^n c^n d^n \mid n \geqslant 0 \}$

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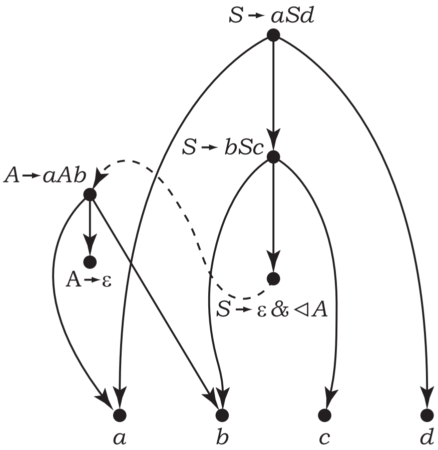

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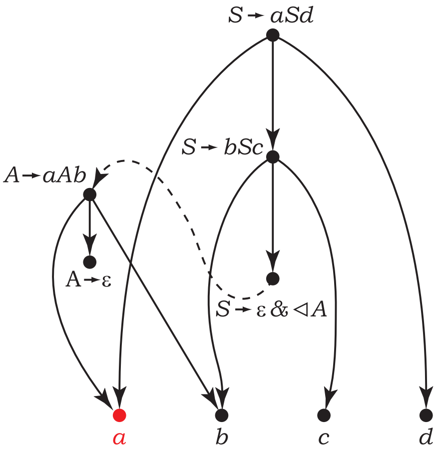
$\vdash [a, \langle \varepsilon \rangle a]$ *(axiom)*



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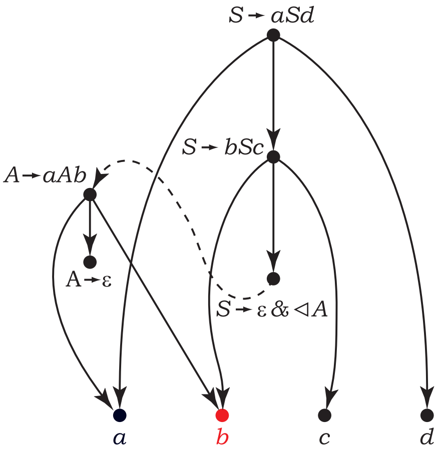

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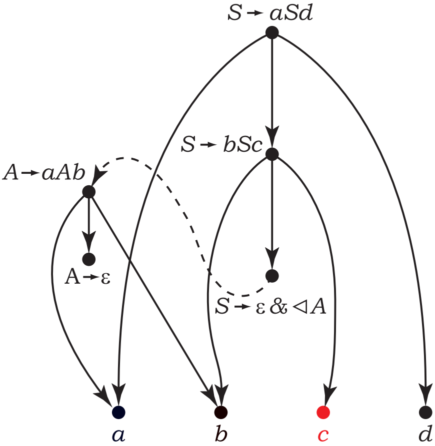

Example revisited

Grammar for $\{ a^n b^n c^n d^n \mid n \geqslant 0 \}$

$$S \rightarrow aSd \mid bSc \mid \varepsilon \,\&\, \lhd A$$
$$A \rightarrow aAb \mid \varepsilon$$

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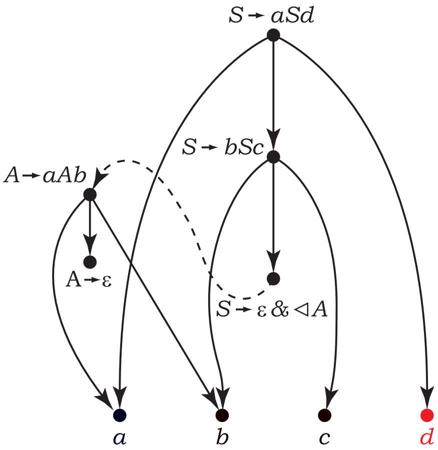

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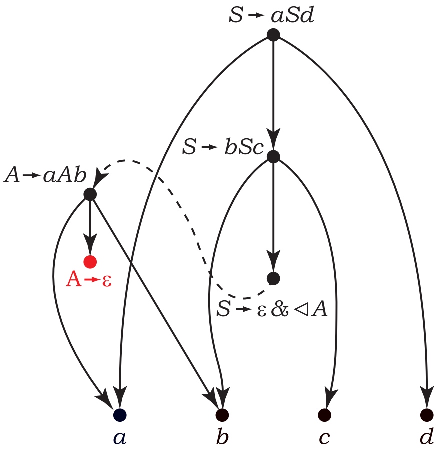

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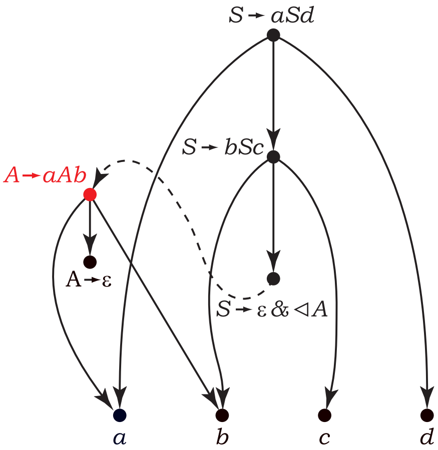

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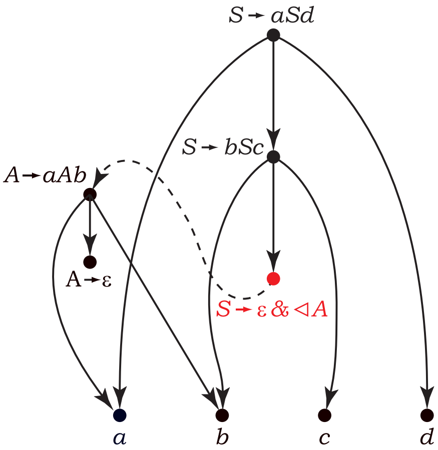

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$$[A, \langle\varepsilon\rangle ab] \vdash [S, \langle ab\rangle\varepsilon] \qquad (S \rightarrow \varepsilon \,\&\, \lhd A)$$

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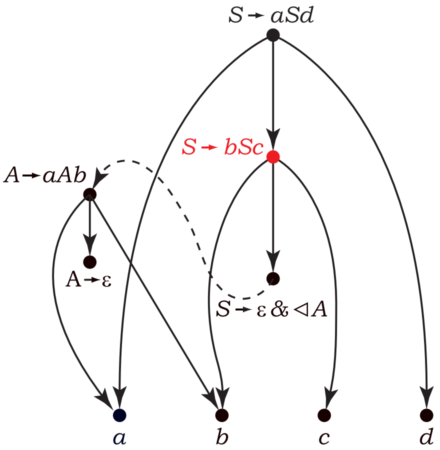

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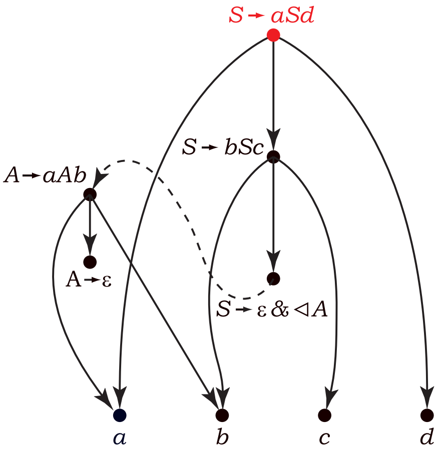

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$[b, \langle a\rangle b], [S, \langle ab\rangle\varepsilon], [c, \langle ab\rangle c] \vdash [S, \langle a\rangle bc]$	$(S \to bSc)$
$[a, \langle\varepsilon\rangle a], [S, \langle a\rangle bc], [d, \langle abc\rangle d] \vdash [S, \langle\varepsilon\rangle abcd]$	$(S \to aSd)$

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Theorem

The two definitions are equivalent.

Complexity of parsing

Theorem

Every grammar with contexts can be transformed to have rules

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- Any negative results?

$$\mathcal{L}(\text{Conj}) \subseteq \mathcal{L}(\text{Conj+contexts}) \subseteq \mathrm{DSPACE}(n) \subseteq \mathrm{NSPACE}(n)$$