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Around the physical Church-Turing thesis

(cellular automata, formal languages and the principles of quantum theory)

Gilles Dowek

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Why a tutorial around the Church-Turing thesis?

A topic that has evolved recently

Connects computability, formal languages, algebra, cellular automata, information theory, communication, ...

Participates to reconsider one of the “postulates” of Computer Science: the independence from hardware: Finally, we may have something to learn from physics

Leads to reconsider the language in which science is written: Computer science transforms all science, but how?

What is there in this tutorial?

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Today

I. The Church-Turing thesis and its various forms

II. Computability beyond natural numbers

III. Gandy's theorem

Tomorrow

IV. The Galileo thesis

V. The principles of quantum theory

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I. The Church-Turing thesis and its various forms

A common situation

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A notion is first understood informally, then precisely defined

E.g. first the notion of a real number, then Cauchy and Dedekind

First the notion of an algorithm, then Herbrand-Gödel, Church, Turing, ...

An uncommon situation

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A discussion: does this definition captures the informal notion?

No such discussion for the notion of a real number

Even for the notion of a distance and of orthogonality based on an inner product

An then a thesis: it does

A possible explanation: History

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First a wrong definition: primitive recursion

Then the Ackermann function

Then the correct? definition

A reasonable doubt

The status of the Church-Turing thesis

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What does it mean for this thesis to be true?

Cannot be proved as it uses an informal notion

Cannot be experimentally tested not a thesis about nature

Could be falsified if anyone came with *ack2*

Yet needs a consensus about the (informal) computability of *ack2*

Three ways to make it a thesis

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Replace the informal notion of computability by another

State some axioms that implicitly define the notion of an algorithm so that the thesis could be proved (remember the real numbers): the algorithmic form of the thesis (Gurevich-Deshowitz)

Analyze the way human mind/brain computes: the psychological form of the thesis (Turing)

Analyze the way a machine computes: the physical form of the thesis (Gandy)

What is a machine?

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Any physical system equipped with a communication protocol

Prepare a physical system by choosing some parameters (input) $a = \langle a_1, ..., a_n \rangle$ and measure others (output) $b = \langle b_1, ..., b_p \rangle$

The physical Church-Turing thesis

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A machine (= physical system + protocol) defines a relation

$a \; R \; b$ if b is a possible output for the input a

Realized by this machine

All relations realized by a machine are computable (*i.e.* r.e.)

A statement about nature

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Easy to imagine alternative universes where this thesis holds

and ones where it does not (accelerating machines, infinite parallelism, ...)

Again: status

According to some (*e.g.* Deutsch) it is a principle of physics

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According to others (*e.g.* Gandy) it must be derived from more fundamental principles of physics

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II. Computability beyond natural numbers

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Computable functions and relations: over $\mathbb{N}$

Physics: parameters and measures, states of a system, ...

Extend computability to sets other than $\mathbb{N}$?

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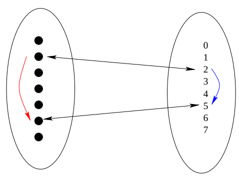
Indexing

Use computability over natural numbers

Transport it to an other set S by an indexing

A bijection between S and (a subset of) $\mathbb{N}$



$$\ulcorner f(x) \urcorner = \hat{f}(\ulcorner x \urcorner)$$

An objection

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Montague (1960): computability depends on the choice of the indexing

The Devil's function

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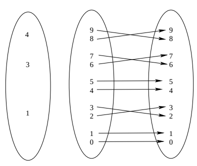
U undecidable set that does not contain 0, D mapping

- $2n$ to $2n+1$ and $2n+1$ to $2n$ if $n \in U$
- $2n$ to $2n$ and $2n+1$ to $2n+1$ otherwise



Non computable involution from $\mathbb{N}$ to $\mathbb{N}$ ($D(0) = 0$, $D(1) = 1$)

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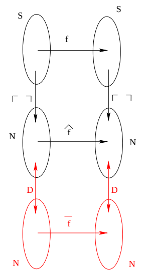
Two indexings of the same set



$D \circ \ulcorner . \urcorner$ also an indexing of S

Same computable functions?

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No

$\hat{f}(n) = 0$ if n is even and 1 otherwise computable

Associated f from S to S computable with respect to $\ulcorner . \urcorner$

But $\overline{f} = D \circ \hat{f} \circ D$ non computable

$\overline{f}(2n) = 1$ iff $D(\hat{f}(D(2n))) = 1$ iff $\hat{f}(D(2n)) = 1$ iff $D(2n)$ odd iff $n \in U$ and U undecidable

Thus f is non computable with respect to $D \circ \ulcorner . \urcorner$

No absolute computability, no absolute Church-Turing thesis

An answer to this objection: stability

We should not be interested in sets, but in sets equipped with operations $\langle \mathbb{N}, S\rangle$, $\langle \mathbb{N}, +, \times\rangle$, $\langle L, ::\rangle$, $\langle L, @\rangle$, ...

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Operations must be computable

(Rabin, 1950): Restrict to admissible indexings: those that make the operations computable

An example: $\langle \mathbb{Q}, +, -, \times, /\rangle$

An admissible indexing: $\ulcorner (-1)^s n/p \urcorner = (s; (n; p))$

where $(n; p) = (n + p)(n + p + 1)/2 + p$

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Any admissible indexing i defines same computability as $\ulcorner . \urcorner$

Computable functions $\overline{+}, \overline{-}, \overline{\times}, \overline{/}$ such that

$$i(x + y) = i(x) \,\overline{+}\, i(y)$$

etc.

Computing $i(x)$ from $\ulcorner x \urcorner$

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Compute $s = fst(\ulcorner x \urcorner)$, $n = fst(snd(\ulcorner x \urcorner))$,
$p = snd(snd(\ulcorner x \urcorner))$

then

$$\underbrace{i(-1)}_{s \text{ times}} \;\overline{\times}\; (\underbrace{i(1) \overline{+} \ldots \overline{+} i(1)}_{n \text{ times}}) \;\overline{/}\; (\underbrace{i(1) \overline{+} \ldots \overline{+} i(1)}_{p \text{ times}}) = i(x)$$

Thus: a computable function h such that $i = h \circ \ulcorner . \urcorner$

and also a computable function k such that $k \circ i = \ulcorner . \urcorner$
$(k(y) = \mu a(h(a) = y))$

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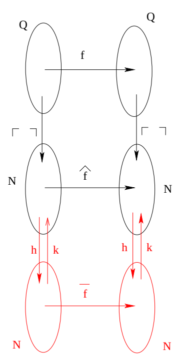

$\overline{f}$ computable iff $\hat{f}$ computable, f computable iff f computable

Stable structures

A structure is stable if all admissible indexings define the same set of computable functions

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$\langle \mathbb{Q}, +, -, \times, / \rangle$ stable

Finitely generated structures are stable

All elements built from a finite subset with operations of structure

Same proof

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Finite extensions of the field $\langle \mathbb{Q}, +, -, \times, / \rangle$

Finite dimensional vector spaces (over stable fields) are stable

Abstracting on admissibility (Boker-Dershowitz, 2006)

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An example: L finite sequences in a finite alphabet

$\langle L, @\rangle$ finitely generated hence stable

χ characteristic function of terminating programs

Rabin: if $@$ is i-computable then χ is not

Boker-Dershowitz: $@$ and χ cannot be computable for the same i

The set $\{@, \chi\}$ non computable in an absolute sense

The Boker-Dershowitz theorem

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C set of computable functions for an admissible indexing of $\langle L, @\rangle$ is maximal

ϕ not in C and i-computable, then i not admissible and $@$ not i-computable

Conclusion: cheating on the encoding to gain a function would cause damage about well-known computable functions

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The physical Church-Turing thesis

(as any thesis about nature)

Can be challenged experimentally (search for experimentally testable consequences)

Can be proved (or refuted) from other physical hypotheses

Can be shown to be consistent with (independent from) other physical hypotheses

Newtonian theory

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The physical Church-Turing thesis is not a consequence of the principles of Newtonian theory

Its negation is consistent with the principles of Newtonian theory

E.g. accelerating machines (just double the clock speed at the end of each cycle)

Gandy's theorem

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A proof of the physical Church-Turing thesis

From three other hypotheses

1: The homogeneity of space and time

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Space is homogeneous

Time is homogeneous

2: Nothing can travel faster than light

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Faster and faster trains (1981: 380 km/h, 2007: 574.8 km/h)

Will it ever stop?

Yes: no train will ever be faster than c

In particular

Information has a finite velocity

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For physicists: An event can influence a system only after a time that is proportional to its distance to the system

3: Information has a finite density

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Larger and larger flash drive capacity (2000: 1 Gb, 2009: 256 Gb)

Will it ever stop?

For trains yes: c

And for flash drives?

A principle

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The amount of information that can be stored in 1 cm^3 is bounded by some constant

For physicists: The number of possible states of a region of 1 cm^3 is bounded by some constant

The origin of the idea

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Gandy (1980)

Bekenstein (1981) gives a bound

Dual hypotheses

Two violations of the physical Church-Turing thesis

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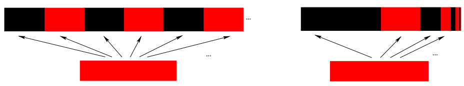

In a given time information can travel through a region populated by a finite amount of information

Proving the physical Church-Turing thesis

Discretize space and time in an arbitrary way

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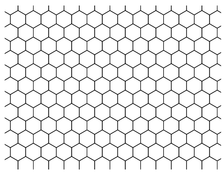

Each cell has a finite state space (3.)

Quiescence: at origin, all but a finite number of cells quiescent

The state of a cell depends of the state of a finite number of cells the previous time step (2.)

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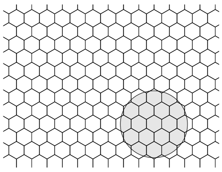

Local evolution function:
homogeneous in time and space (1.) (cellular automaton)
finite hence computable, thus global evolution computable

Criticizing Gandy’s hypotheses

Gandy’s hypotheses can be / have been criticized

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Do not hold in the Newtonian theory

Do not hold in the quantum theory (superposition)

The Newtonian theory

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Bounded velocity of information?

Gravitation has an instantaneous effect on all the Universe

Already criticized at the time of Newton

A weak point of Gandy's hypotheses or of Newtonian theory?

The Newtonian theory

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Bounded density of information?

No, a caliper encodes a real number (infinite sequence of bits)

But ...

a caliper only gives you three significant digits

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A principle of the finiteness of the number of significant digits

Added to Newtonian theory

The classical expression of the hypothesis (3.)

Empirical evidence?

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No instantaneous network

No USB flash drive with an infinite capacity (so far ...)

Just an example?

Gandy's hypotheses may need to be refined

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Yet, an example of objective properties of nature that explain the physical Church-Turing thesis

Don't miss tomorrow's talk

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Galileo, quantum, and more