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Around the physical Church-Turing thesis

(cellular automata, formal languages and the principles of quantum theory)

Gilles Dowek

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Previously on *Around the physical Church-Turing thesis*

I. The Church-Turing thesis and its various forms

II. Computability beyond natural numbers

III. Gandy's theorem

Today

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IV. The Galileo thesis

V. The principles of quantum theory

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IV. The physical Church-Turing thesis as an explanation of the Galileo thesis

The Galileo thesis

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Philosophy is written in this vast book, which continuously lies open before our eyes (I mean the Universe). But it cannot be understood unless you have first learned the language and recognize the characters in which it is written. It is written in the language of mathematics.

Why?

A mystery

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Einstein: The eternal mystery of the world is its comprehensibility

Wigner: The enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no explanation for it

A few attempts

- God is a mathematician
- The mathematical concepts are built by abstracting from empirical ones

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- The scientists select mathematizable phenomena
- The scientists approximate phenomena to make them mathematizable
- We are part of nature: our mathematical concepts are natural

A few attempts

- God is a mathematician why?
- The mathematical concepts are built by abstracting from empirical ones ellipses?

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- The scientists select mathematizable phenomena changes the scope, not the nature of the problem
- The scientists approximate phenomena to make them mathematizable and why can they be approximated?
- We are part of nature: our mathematical concepts are natural do we understand better from the inside?

A particular case

Slide 57 Physically realized relations can be expressed by a proposition in the language of mathematics

Physically realized relations

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An experiment where one prepares a physical system by choosing parameters $a = \langle a_1, ..., a_n \rangle$ and measures others $b = \langle b_1, ..., b_p \rangle$

This experiment (= physical system + protocol) defines a relation

$a \; R \; b$ if b is a possible result of the measures of the system prepared with a

The Galileo thesis

Physically realized relations are definable by a proposition in the language of mathematics

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Notice how small this set is: countable

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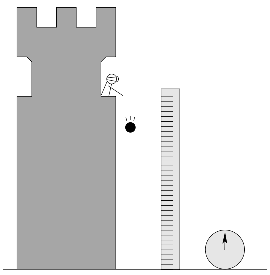

$$x = \frac{1}{2}gt^2$$

Experiments and machines

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Physical system + protocol ... I have seen this before

Yesterday: our definition of a machine

The notions of an experiment and of a machine are co-extensive

The free fall in vacuum

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The Galileo thesis: the relation between t and x is definable by a proposition in the language of mathematics

The physical Church-Turing thesis: the relation between t and x is computable

Both the Galileo thesis and the physical Church-Turing thesis
speak about physically realized relations

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The Galileo thesis: physically realized relations are definable by a proposition in the language of mathematics

The physical Church-Turing thesis: physically realized relations are computable

But

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Computable relations are definable by a proposition in the language of mathematics (even in PA)

Thus ...

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If the physical Church-Turing thesis holds then so does the Galileo thesis

Eliminating the physical Church-Turing thesis

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Gandy's hypotheses $\Rightarrow$ the physical Church-Turing thesis

The physical Church-Turing thesis $\Rightarrow$ the Galileo thesis

Gandy's hypotheses $\Rightarrow$ the Galileo thesis

(Yet: the physical Church-Turing thesis showed the link between Gandy's hypothesis and the Galileo thesis)

Gandy's hypotheses $\Rightarrow$ the Galileo thesis

The efficiency of mathematics to describe the Universe consequence of

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objective properties of the Universe

A property of the Universe or of the theory?

An early objection to Galileo: the Universe is not a book

We write the book in the language of mathematics

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A sixth explanation: the book is written in the language of mathematics because we write it this way

But ...

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Not everything is a theory of the Universe (some sort of experimental adequation with the Universe)

We could have tried to write the book in the language of mathematics and failed

The possibility to write this book still remains to be explained

Which degree of freedom

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for choosing the set of physically realized relations when constructing a theory of the Universe?

Very little

If two theories differ on the set of physically realized relations, then one can be experimentally refuted (in the functional case) or experimentally statistically refuted (in the general case)

A particular case: all realized relations are functional

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Two theories $\mathcal{T}$ and $\mathcal{T}'$

A relation R realized according to $\mathcal{T}$ but not $\mathcal{T}'$

The experiment E realizing R according to $\mathcal{T}$

The relation R' realized by E according to $\mathcal{T}'$

$R \neq R'$

a, b, b', $b \neq b'$ such that $a\ R\ b$, $a\ R'\ b'$

Do the experiment E with a either the result is b: $\mathcal{T}'$ refuted, b': $\mathcal{T}$ refuted, some other value: both theories refuted

The general case

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Either one theory can be refuted or it predicts (among others) a result that never happens (statistical experimental refutation)

A second Galilean revolution

Propositions

$$x = \frac{1}{2}gt^2 \qquad \ddot{x} = g$$

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or algorithms

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fun t -> 0.5 * g * t * t
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Particular case (solutions of differential equations)

More general (biological processes)

V. The physical Church-Turing thesis in quantum theory

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(joint work with Pablo Arrighi)

As for Newtonian theory

Slide 75 Is the physical Church-Turing thesis a consequence of the principles of quantum theory?

No (Nielsen)

Slide 76 States of a quantum system are elements of a vector space

And transformations are unitary linear maps (isometries)

No (Nielsen)

A system with an infinite dimensional state space

An orthonormal basis of this space $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots$

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E undecidable

An operator U mapping $\mathbf{e}_{2i}$ to $\mathbf{e}_{2i}$ and $\mathbf{e}_{2i+1}$ to $\mathbf{e}_{2i+1}$ if $i \in E$ and $\mathbf{e}_{2i}$ to $\mathbf{e}_{2i+1}$ and $\mathbf{e}_{2i+1}$ to $\mathbf{e}_{2i}$ otherwise

We have $\langle \mathbf{e}_{2i} | U \mathbf{e}_{2i} \rangle = 1$ if $i \in E$ and 0 otherwise

No (Nielsen)

Quantum theory does not forbid the existence of an evolution described by the operator U

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Although there seems to be no such evolution

Quantum theory is too liberal

Generalizing Gandy's hypotheses to the quantum case

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Finite velocity of information

Compatible with the EPR paradox?

No accessible information can be communicated faster than light

Sufficient for Gandy's proof to generalize?

Generalizing Gandy's hypotheses to the quantum case

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Finite density of information

Finite state space?

If two states $|\phi\rangle$ and $|\phi'\rangle$ then all $\lambda|\phi\rangle + \mu|\phi'\rangle$ are states

Plus: the state of the whole space is more than the collection of the states of each cell (entanglement)

Generalizing Gandy's hypotheses to the quantum case

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Plus: scalars may encode an infinite amount of information

$\lambda = \sum h(i)/3^i$ and μ such that $|\lambda|^2 + |\mu|^2 = 1$

$\lambda|\phi\rangle + \mu|\phi'\rangle$ is a possible state of the system

Finite density of information

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For every measurement, there is a finite number of possible outcomes

Boils down to: the state space of a finite region of space a finite dimensional state space

Finite density of information

Scalars may encode infinite amount of information

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Restrict to a countable field: finite extension of the field of rationals, *e.g.* $\mathbb{Q}(i, \sqrt{2})$

From finite velocity of information to locality

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Causality (finite velocity of information): the state of a region of space A at time $t + T$ is a function of the state of the (whole) area of radius cT around this region at time t

Locality: The state of the whole space at time $t + T$ by applying a composition of local linear maps to the state of the space at time t

Local linear maps: extension to an infinite dimensional vector space of a linear map defined on a finite dimensional space

Slide 85 Arrighi-Nesme-Werner (2010): Causality + unitarity $\Rightarrow$ locality

Putting the pieces together

The sate space of a cell is a finite dimensional space (3.)

Slide 86 The (infinite dimensional) state space of the whole space: iterated tensor product of finite dimensional vector spaces

On *e.g.* $\mathbb{Q}(i, \sqrt{2})$ (3.)

Vectors can be indexed

Putting the pieces together

The evolution is described by a local linear map (2. + ANW)

The local map is independent of space and time (1.)

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Finite functions are computable

Linear maps defined in a finite dimensional space are computable (matrix multiplication)

Local linear maps are computable

The overall evolution function is computable

Indexing?

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Computability is stable on iterated tensor products of a finite dimensional vector space

Hard-wiring Gandy's hypotheses
in the quantum theory formalism?

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Infinite dimensional vector spaces: only iterated tensor products of a finite dimensional vector space

Smaller field

Evolutions: only compositions of local linear maps

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Time to end

What I did not speak about

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A hidden assumption in Gandy's proof (and our generalization) space and time are absolute (from cellular automata to graphs)

Computability over the real numbers (differential equations as a programming language)

Hypercomputability (adiabatic relaxation, black holes, ...)

Complexity (physical P $\neq$ NP?)

The physical Church-Turing thesis

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Challenges the way science is written (algorithms, infinite dimension vector spaces, scalars, ...)

Proposes new principles to physics (boundedness of the density of information)

But also explains the enormous usefulness of mathematics in the natural sciences

The physical Church-Turing thesis

A principle invented in logic and computer science

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That goes beyond logic and computer science