Shore's computational reverse mathematics

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Since its inception in the 1970s, the programme of reverse mathematics has been remarkably successful in proving the equivalence of many theorems of classical mathematics to subsystems of second order arithmetic. Since it allows us to characterise precisely what axioms are required in order to prove a particular theorem, reverse mathematics provides a valuable tool for assessing attempts to give foundations for mathematics. When committing to a particular foundational system, reverse mathematics lets us know just what we are giving up. It also tells us when a proponent of such a system employs mathematical resources that she is not entitled to, as they go beyond what her preferred foundation can prove.

In two recent papers, Richard Shore [2010, 2011] proposes an alternative approach to reverse mathematics, based on notions from computability theory. We suggest that while Shore's analysis is mathematically promising, it appears to suffer from several defects if understood as an attempt to meet the foundational goals of reverse mathematics within a computational framework.

1 Computable reverse mathematics

Friedman [1975] showed that much of classical mathematics could be formalised and proved in the setting of second order arithmetic. It has been claimed (for example by Simpson [2009]) that this has philosophical ramifications, as many of the important subsystems of second order arithmetic correspond to philosophically motivated foundations of mathematics, such as Bishop's constructivism and the predicativity of Weyl and Feferman.

Computability theory has long provided important tools for the practicing reverse mathematician. Take a set $\mathcal{C} \subseteq \mathcal{P}(\omega)$ which is closed under Turing reducibility and recusive joins. We call such an $\mathcal{C}$ a *Turing ideal*. An ω-model with $\mathcal{C}$ as its second order part is a model of the usual base theory for reverse mathematics, RCA_0 (as well as its counterpart with full induction, RCA). By constructing Turing ideals of an appropriate sort, we can thus prove nonimplications between statements. Consider two formulae $\varphi(X, Y)$ and $\psi(X, Y)$ such that $\forall X \exists Y \varphi(X, Y)$ and $\forall X \exists Y \psi(X, Y)$ are both Π^1_2. We construct a Turing ideal $\mathcal{C}$ such that for every $X \in \mathcal{C}$ there is a $Y \in \mathcal{C}$ such that $\psi(X, Y)$, but there is an $X \in \mathcal{C}$ for which $\varphi(X, Y)$ is not satisfied by any $Y \in \mathcal{C}$. We thus conclude that $\forall X \exists Y \psi(X, Y)$ doesn't imply $\forall X \exists Y \varphi(X, Y)$ over RCA.

Shore proposes taking this use of computability theory a step further and basing a new approach to reverse mathematical analysis on recursion theory, rather than proof theory. In place of the usual relations employed in reverse mathematics—provability and logical equivalence over a base theory—he offers the notions of *computable entailment* and *computable equivalence*.

Definition 1.1. Let $\mathcal{C}$ be a Turing ideal, and let φ be a sentence of second order arithmetic. $\mathcal{C}$ *computably satisfies* φ if φ is true in the ω-model whose second order part consists of $\mathcal{C}$. A sentence ψ *computably entails* φ, $\psi \models_{\mathcal{C}} \varphi$, if every Turing ideal $\mathcal{C}$ satisfying ψ also satisfies φ. Finally, ψ and φ are *computably equivalent*, $\psi \equiv_{\mathcal{C}} \varphi$, if each computably entails the other. (The extension of these definitions to sets of sentences is obvious.)

Unlike other computational accounts of reverse mathematics, such as Dean and Walsh [2011], Shore's approach is revisionary: computable entailment and equivalence do not coincide with their proof theoretic counterparts. Failure of computable entailment, $T \not\models_{\mathcal{C}} \varphi$, implies $T \nvdash \varphi$, and of course $T \vdash \varphi$ implies $T \models_{\mathcal{C}} \varphi$, but neither of the converses of these statements holds. In other words, this is not just another way of looking at what we already have in reverse mathematics, but another game altogether, albeit a closely related one.

2 The unity of mathematics

Because reverse mathematics is pursued in second order arithmetic, it is limited to the analysis of countable or countably representable mathematics. Shore presents its natural generalisation to uncountable structures as one of the major benefits of his view. Prima facie this seems like a reasonable claim: all that's required to generalise his view is plugging in an appropriate notion of computation into the definition of computable entailment. The question thus becomes: which notion of computation on uncountable structures is the appropriate one?

It is safe to say that this is a vexed question. Indeed, it is far from clear that there even *is* a single correct notion of uncountable computability. Two cases naturally distinguish themselves: computation on the reals, and computation on structures of arbitrary cardinality. For the latter case, Shore proposes α-recursion, and much of his [2011] is devoted to proving that analogues of ACA and WKL can be defined and function as expected in inital segments of the constructible universe L_κ for arbitrary uncountable cardinals κ.

In the case of computation on $\mathbb{R}$, it seems natural to assume that we don't have a wellordering of the structure, and thus a different notion of computability may be appropriate—Shore suggests E-recursion, Blum-Shub-Smale computability and Kleene recursion in higher types. Rather than offer a preferred candidate, he argues that the very practice of computable reverse mathematics may allow us to answer the question raised at the beginning of this section.

However, even if this optimistic view is correct, we still find that Shore's generalisation has left us with three distinct cases, each with their own notion of computation: the natural numbers, and countable structures generally; subjects such as analysis where the basic underlying set is the reals; and finally, combinatorial or algebraic study of structures with arbitrarily high uncountable cardinalities.

One of the appeals of reverse mathematics is its unificatory power: it demonstrates how, when reformulated in a single formal system, theorems from many different branches of mathematics turn out to be equivalent. Prima facie this should apply to cardinality considerations also: any generalisation of reverse mathematics to encompass the analysis of uncountable structures should preserve the comparability of theorems.

In other words, Shore's proposed method of generalising reverse mathematics breaks the generality of reverse mathematical analysis. It does not provide a single framework within which we can compare the theorems of different branches of mathematics.

3 The coarse graining problem

Computable entailment is weaker than deductive entailment, since it amounts to restricting our attention to a particular class of models. This collapses many distinctions present in the proof theoretic case. For instance, the standard natural numbers $\omega = \{0, 1, 2, \ldots\}$ satisfy the induction scheme for all definable predicates in the language of second order arithmetic. As a result, systems with only restricted induction and their counterparts with the full Z_2 induction scheme are computably equivalent: RCA_0 and RCA have the same ω-models, as do ACA_0 and ACA, and so on.

Since Friedman [1976] systems with restricted induction have been used as the main yardsticks in reverse mathematics. This is not an accident, and many consider these systems more natural. Whether this is the case is not directly relevant here: the key point is that this is a distinction worth preserving. In support of this claim, consider the following facts. Firstly, there are important conservativity theorems for these systems. ACA_0 is both proof theoretically and model theoretically conservative over first order PA. RCA_0 is similarly conservative over $\mathrm{I}\Sigma_1$. Neither result holds for the analogous systems satisfying full induction. Further, the systems with full induction also have greater proof theoretic ordinals than the restricted versions. Finally, ACA proves the consistency of ACA_0, and RCA proves the consistency of RCA_0. That these systems are computably equivalent should make us worry that, whatever the other mathematical virtues of Shore's computational reverse mathematics, its basic notions are too coarse grained and fail to respect important distinctions between systems.

Since computable entailment is defined in an analogous way when we move to uncountable structures, it seems reasonable to suggest that similar issues will occur there, regardless of which notion of computability we employ.

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