Incompleteness, Church's Thesis, and Mathematical Knowledge

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1 Incompleteness and Mathematical Knowledge

Gödel's incompleteness theorems are deemed to reveal something fundamental about the limits and potentialities of our mathematical knowledge. They show that for any given consistent recursively axiomatised system of suitable arithmetical strength, there is a statement which is true but not formally provable in that system–i.e., *absolute provability* outstrips provability in a formal system.

However, if a statement that *says* of itself that it is formally unprovable in a formal system is provably so, then this statement is evidently true, and should be added to the initial formal system. Hence, a natural way of attempting at capturing mathematical knowledge in the light of the incompleteness theorems would be to extend a formal system for mathematics by adding statements that we recognise as correct as new axioms, to form increasingly stronger systems of axioms that we also recognise as correct.

This route was first pursued by Turing [8], who introduced the notion of *ordinal logics*–where *logics* denotes a collection of formal systems–as a way of overcoming the phenomenon of incompleteness of formal theories containing arithmetic, only eight years after Gödel's paper on the incompleteness theorems. Ordinal logics are built by systematically extending the starting theory by adding at each stage a true but unprovable statement to the effective union of the axiom sets of the preceding theories. The statement that Turing added as an axiom was of the form $\forall x R(x)$ (with R primitive recursive), equivalent to the statement formalising the consistency of the formal system. The iterations are counted by effectively associating formal systems with ordinals, and collecting all earlier theories into a single new theory at every limit ordinal. In order to deal with limit stages effectively, a system of notations for ordinals is needed; in this case, it is used Kleene's $\mathbf{O}$ for constructive ordinals.

Gödel proved that the incompleteness theorems apply also to any such extensions of formal systems. However, they do not refute the possibility that some specific formal system, or collection thereof, may capture all true mathematical sentences (hence, if complete, such system will not be recursively axiomatisable). In particular, since the passage from one system to the following one in Turing's ordinal logics is itself effective, the iteration can be carried on into the transfinite. So the question arises as to whether some ordinal logics associated with a constructive notation a for ordinals is complete with respect to Π_1-statements. This amounts to asking whether it is the case that given any Π_1-statement, if it is true, then it is provable in some ordinal logics associated with a notation $a \in \mathbf{O}$.

Turing's main result was a Π_1-completeness proof for his ordinal logic–i.e., the proof that it is always possible to choose an $a \in \mathbf{O}$ with $|a| = \omega + 1$ such that if a Π_1-statement ϕ is true, then it is provable in the system associated with a. However, Π_1 is a low level of complexity, so the question arises as to what is the degree of completeness of logics associated with notations.

2 Overcoming Incompleteness

Feferman [1] strengthened Turing's result and proved Π_2-completeness and full completeness by constructing hierarchies of theories by iterating uniform reflection principles (called *transfinite recursive progressions of formal systems* to differentiate them from Turing's ordinal logics). Informally, a *reflection principle* for a formal theory T is a statement about T that is a consequence of T being sound. More precisely, for every n, $\mathrm{REF}^n(\phi)$ is a Π_{n+1}-sentence formalising the statement "T is Σ_n-sound" (where ϕ is a Σ_1-formula defining the axioms of T).

When T is extended by adding $\mathrm{REF}^n(\phi)$ as a new axiom, this yields an *extension by n-reflection* of T; ϕ being a Σ_1-formula is enough to ensure that any extension by reflection of T is logically stronger than T. An *iterated n-reflection extension* of a theory T_0 can be intuitively thought of as a theory in a sequence $T_0, T_1, \ldots, T_\omega, T_{\omega+1}, \ldots, T_\alpha, \ldots$ of theories, where $T_{\alpha+1}$ is an extension by n-reflection of T_α, and where T_λ, for limit ordinals λ, has as its axioms the union of the axioms of earlier theories.

Feferman's completeness result is an application of Shoenfield's completeness theorem for the recursive ω-rule to *families* of theories–i.e., assignments of theories T_a (that extend PA) to every a in some set B of natural numbers. Instead of a set of proofs in Shoenfield's formulation, this application yields a set of indexes of theories:

> **Shoenfield's completeness theorem:** For any family of theories which is closed under the recursive ω-rule, and any true arithmetical sentence ϕ, there is a $b \in B$ s.t. $T_b \vdash \phi$.

Informally, Shoenfield's completeness theorem allows us to infer at limit stages that given $\phi(x)$, if it is provable that there is a Turing machine that proves $\phi(x)$ for all x, then $\forall x\ \phi(x)$. At the next stage, it is possible to add the reflection principle for the union of the axioms of earlier theories and iterate this process for each successor ordinal until the next limit ordinal.

Three further main steps allow to prove full completeness. First, Feferman proved Turing's Π_1-completeness theorem in terms of reflection extensions. He was then able to prove Π_2-completeness:

> **Π_2-completeness theorem:** For any true Π_2-sentence ψ and base theory T there is a 1-reflection sequence of length $\omega + 1$ in which the last theory proves ψ.

The Π_2-completeness theorem implies that ω-reflection progressions are locally closed under the recursive ω-rule.[1] By the modified Shoenfield's completeness theorem it follows:

> **Feferman's completeness theorem:** Given an ω-reflection progression, an ω-reflection sequence can be extracted such that every true arithmetical sentence is provable in some theory in the sequence.

3 Transfinite Progressions and Church's Thesis

Despite such striking result, the question of *absolute undecidability*–the question of whether there are mathematical statements which are unprovable relative to any justified set of axioms–is still taken to be open. This is because there are reasons to be sceptical about the extent to which Feferman's completeness theorem can illuminate an account of mathematical knowledge:

[1] For the purpose of this abstract, progressions can be thought of as a special case of families.

1. we don't know, on the basis of the theorem, whether any given arithmetical sentence ϕ (whose truth value is not yet known) is true, because there is no effective procedure that tells us whether ϕ is a member of the set of theorems provable in the reflection sequence of length less than ω_1^{CK} shown to be complete for arithmetical truth;

2. the structure of the tree of progressions induced by Kleene's **O** is such that there is no obvious way to justify in advance the choice of the notation at limit stages;

3. the construction of reflection sequences relies on non-standard definitions of the axioms of a formal system.[2]

I argue that a distinction made by Kreisel can help assessing the philosophical import of Feferman's completeness theorem by formulating an interesting conjecture. Kreisel [6] distinguishes *m-effective* and *h-effective* definitions. The former captures the notion of *mechanically performable instruction* by a machine; Turing's analysis of this notion shows that it is *intensionally* equal to some program for an idealised computer.

The latter captures the notion of *humanly performable instruction.* While machine-computability is a precisely defined and well-understood notion, and Church's Thesis holds for it, no satisfactory explication of human-computability is available. The idea is to capture a notion which does not constitute an idealisation too far removed from ordinary mathematical practice: beyond computing in a finite number of steps, human-computability uses a language and a list of axioms and inference rules that are fixed and finite at any specific point in time. Abstracting away from limitations of time, on the other hand, machine-computability is still bounded to computation on the basis of a finite list of instructions. With respect to human-computability, instead, it could be argued that the possibility of new axioms is always open, so the list of axioms that an ideal community of mathematicians could compute in an infinite time is potentially infinite.

According to Kreisel, the truth value of Church's Thesis for human-computability determines the limits of mathematical reasoning: h-effective definable functions can be seen as the analogue of provable theorems. In this sense, the notion of human-computability could constitute a rigorous explication of the notion of human provability which is not relativised to any given formal system.

I suggest that human-computability of $\phi(x, y)$ could be cashed out as follows:

$$\Box\forall x\exists y\ \Box\phi(x, y),$$

where the modal operator $\Box$ (added to the language of PA with $S4$ axioms) describes the informal notion of provability in arithmetic, and ϕ is a relation holding between y and x. If this formalisation is correct, then we could ask whether Church's Thesis holds for it.

If so, the following would hold:

$$\phi(x, y) \text{ is } h\text{-effectively computable iff } \Box\forall x\exists y\ \Box\phi(x, y).$$

Call this *Epistemic Church's Thesis* (ECT).

As yet, there is no conclusive argument for ECT or its negation; however, Horsten&Leitgeb [5] showed that if ECT holds, then there are absolutely undecidable sentences. Thus, if a convincing case for this suggestion could be made, this could shed further light on the relevance of transfinite progressions for an account of the limits of mathematical knowledge.

[2] See [3].

References

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