Mathematics without actual infinity

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May 8, 2012

1 Introduction

The main motivation of the work reported here is searching the borderline of finitely meaningful mathematics, as it is called by Jan Mycielski [19], mathematics without actual infinity.

We give here a short survey of selected topics related to logical research on foundations of mathematics without actual infinity. We start with a short survey of old ideas. Then we discuss various methodologically motivated approaches to the problem. Finally we report last results of the author on sl–semantics and first order logic.

2 Some older ideas

2.1 Actual and potential infinities in foundation of mathematics

The old Aristotelian opposition of potential and actual infinities was popularized in modern times by David Hilbert in [6]. Actually infinite set is just an infinite collection given as one entity. The same thing considered as only potentially infinite is a finite collection which can be arbitrary enlarged according to a given rule. What a difference for doing mathematics. Let us illustrate on the example.

In 1930 Kurt Gödel published the first of his crucial papers [1][1] He proves there the completeness theorem for first order logic. Frequently people ask how was it possible to prove this theorem before semantics obtained its precise

[1]The new edition with English translation can be found in [2].

mathematical definition – what happened not earlier than in 1933 in the work by Alfred Tarski [21].

The main part of the proof, for the case of a consistent formula when a model has to be constructed, he concludes as follows:

"... it follows by familiar arguments that in this case there exists a sequence of satisfying systems $S_1, S_2, \ldots, S_k, \ldots$ (S_k being of level k) such that, after S_1, each contains the preceding one as a part. We now define in the domain of *all* integers ≥ 0 a system S ..."[2]

The system S is then defined by explicite defininition of all relation in the way equivalent to saying that $S = \bigcup_k S_k$.

Finitistic mathematics, as not using actual infinity, is safe.

For the finite models – according Hilbert's claim – we can use intuitive arguments. In the infinite case such arguments are unsafe. Then Gödel tried to use semantical arguments only in the finite case.

2.2 From potential to actual infinity and back

As we know later on in the second half of twenty century foundations of mathematics absorbed actual infinity by interpretation of in axiomatic set theory. It was motivated mainly by the style of use mathematics in physics. However, at the end of twenty century computational way of thinking and computational motivations started to be much more important (also in physics). So again we – returning to these old motivations – we ask which mathematical notions can be interpreted without reference to actual infinity. This time not for reforming mathematics, but only for separating its part which is finistically meaningful.

2.3 FM–domains

Actually infinite domain of natural numbers is the set $\{0, 1, 2, \ldots\}$.

As an explication of potentially infinite domain of natural numbers we mean the family of finite approximations of the actually infinite domain. This is the following family:

[2]This is the English translation from [2], p. 117. The German original follows: "... so folgt nach bekannten Schlußweisen, daß es in diesem Fall eine Folge von Erfüllungssystemen $S_1, S_2, \ldots, S_k, \ldots$ (S_k von k–ter Stufe) gibt, deren jedes folgende das vorhergehende als Teil enthält. Wir definieren jetzt im Bereich *aller* ganzen Zahlen ≥ 0 ein System S ...", [2], p. 116.

$$\begin{aligned}&\{0\},\\&\{0,1\},\\&\{0,1,2\},\\&\{0,1,2,3\},\\&\vdots\end{aligned}$$

Therefore, having an arithmetical model $M = (N, R_1, \ldots, R_s)$, we define its potentially infinite version as the family $FM(M) = \{M_1, M_2, M_3, \ldots\}$, where $M_n = (N_n, R_1^{<n}, \ldots, R_s^{<n})$, $N_n = \{0, 1, \ldots, n-1\}$ and $R_i^{<n}$ is the restriction of R_i to the set $\{0, 1, \ldots, n-1\}$.

2.4 Analysis without actual infinity of Mycielski

Jan Mycielski in his paper "Analysis without actual infinity" [19] considers interpretation of basic notions of mathematical analysis which allows finite interpretations. He allows finite as well as infinite interpretations. His theory has the property of being locally finite, it means that each finite part of the theory has a finite model, but all the axioms are true only in infinite models being a kind of nonstandard universes.

3 FM–representability and algorithmic learnability

From the finitistic point of view the crucial question is: what essentially infinite sets and relations can be described in finite models? The answer given in the paper [10] was that they those relations which are FM–representable.

The first remarks on the relevance of the notion of FM–representability in foundations of mathematics can be found in [11] and [13].[3]

3.1 FM–representability

A relation $S \subseteq N^k$ is FM–represented by an arithmetical formula $\varphi(x_1, \ldots, x_k)$ in $FM(M)$ if for each $a_1, \ldots, a_k \in N$ the following two conditions hold:

1. $S(a_1, \ldots, a_k)$ holds if and only if the formula $\varphi(x_1, \ldots, x_k)$ is satisfied by $a_1, \ldots, a_k$ in almost all models from $FM(M)$.

[3] The earlier version of this paper was presented at the first meeting "Computations in Europe" in 2005 and it was published in local proceedings [12].

2. $S(a_1, \ldots, a_k)$ does not hold if and only if the formula $\neg\varphi(x_1, \ldots, x_k)$ is satisfied by $a_1, \ldots, a_k$ in almost all models from $FM(M)$.

A relation $S \subseteq N^k$ is FM–representable in $FM(M)$ if it is FM–represented by some arithmetical formula in $FM(M)$.

When M is the standard model of addition and multiplication then we will say simply *FM–representable* instead of *FM–representable in $FM(M)$*.

The following theorem summarizes the classical results with the FM–representability theorem from [10] (for details see [14]).

Theorem 3.1 (FM–representability) *Let $R \subseteq \omega^k$. Then the following statements are equivalent:*

1. *R is FM–representable.*
2. *R is Δ^0_2 in the arithmetical hierarchy.*
3. *R is recursive with a recursively enumerable oracle.*
4. $\mathbf{deg}(R) \leq \mathbf{0}'$.
5. *R is the limit of some recursive sequence of relations.*
6. *R is algorithmically learnable.*

3.2 Learnability

The idea of algorithmic learnability was formulated independently by Hilary Putnam [20] and Mark Gold [3], [4].

A learning algorithm works infinitely on a given input in steps $n \in N$. On each step it gives an answer "YES" or "NO". After some finite number of changes the answer is stabilizing. The final answer is that which is given on almost all steps.

Mathematical theorems learnable in this sense enlarge our epistemic abilities given by proofs in axiomatic framework. For instance, we learn consistency of some axiomatic theories, such as PA or ZF. We learn of truths of some practically useful theorems, then we act assuming them without having any proof. A good example of such theorem is the claim of hardness of all prime divisors for each given number. On this claim safety of RSA cryptographic is based.

This seems to be a reasonable alternative – for proofs in axiomatic theories – description of obtaining mathematical truths. Of course the method can be applied only to the notions representable in finite models.

3.3 FM–representability by games

A few months ago Artur Wdowiarski proposed the following game interpretation for the case of $S \subseteq N^k$ is FM–represented by an arithmetical formula $\varphi(x_1, \ldots, x_k)$ in $FM(M)$:

We start with a k–tuple $a_1, \ldots, a_k \in N$. Then the players: Player 1 and Player 2 play the game.

Player 1 chooses a number n_1 such that the k–tuple satisfies $\varphi(x_1, \ldots, x_k)$ in M_{n_1}.

Then Player 2 chooses a number $m_1 > n_1$ such that the k–tuple satisfies $\neg\varphi(x_1, \ldots, x_k)$ in M_{m_1}.

Then Player 2 chooses a number $n_2 > m_1$ such that the k–tuple satisfies $\varphi(x_1, \ldots, x_k)$ in M_{n_2}, and so on.

The payer who has no move loses.

Artur Wdowiarski observed that – in this general formulation the game can be finished in one step. However, if we add the requirement that the behavior of the players should be recursive then the game is nontrivial.[4]

When we accept the claim that behavior of human beings should be algorithmic.[5] Then the recursive version of Wdowiarski game gives a good picture of the epistemic game played by human beings — of course in great teams *pro et contra*.

4 sl–semantics

In this section we present shortly the main ideas of [15]. We start with recalling the notion of truth in sufficiently large finite models $\models_{sl}$, defined in [10].

For each class K of finite models and each formula φ:

$$K \models_{sl} \text{ if and only if there is } n \text{ such that for all } M \in K$$

$$\text{if } \mathrm{card}(M) > n \text{ then } M \models \varphi.$$

Originally the definition was given for arithmetical models and it was investigated later in a few papers devoted to finite arithmetics: [18], [16], [9], [7], [8]. However the idea can be generalized in a natural way for any case of potential infinity.

[4]As a matter of fact only the last remark was my contribution. The main idea was formulated by Artur Wdowiarski on my seminar in October 2011.

[5]This claim is discussed shortly in [17].

4.1 π–domains

Let σ be a vocabulary. The set of σ–sentences (closed formulae) is denoted by F_σ. We assume that all the considered vocabularies are purely relational, it means that there are no individual constants and function symbols. Moreover all vocabularies are finite.

Let K be a class of finite models for a given finite relational vocabulary σ. The sl–theory of K, $sl(K)$ is the set of all those sentences from F_σ which are true in almost all models from K, that is

$$sl(K) = \{\varphi \in F_\sigma : \exists k \forall M \in K(\mathrm{card}(M) > k \Rightarrow M \models \varphi)\}.$$

For a class of finite models K and a formula φ we define *truth in the limit* relation $\models_{sl}$ as follows

$$K \models_{sl} \varphi \text{ if and only if } \exists k \forall M \in K(\mathrm{card}(M) > k \Rightarrow M \models \varphi).$$

Thus we can define $sl(K)$ equivalently as

$$sl(K) = \{\varphi \in F_\sigma : K \models_{sl} \varphi\}.$$

Let us observe that if the class K contains only models of cardinality bounded by some natural number n then $sl(K) = F_\sigma$. So sl–theory of this class is in a sense inconsistent.

A class K of finite σ–models is unbounded if and only if for each n there is $M \in K$ such that $\mathrm{card}(M) > n$. The class K will be called *unbounded σ–class* or simply *unbounded class* when a vocabulary will be clear from the context.

Unbounded classes containing for each cardinality at most one (up to isomorphisms) model approximate in a sense an infinite domain. Such classes will be called *potentially infinite domains*, or *pi–domains*, or *π–domains.* When a π–domain K describes an infinite structure in a stronger sense, that is $K = \{M_0, M_1, M_2, \ldots\}$ and for all n, $M_n \subseteq M_{n+1}$, then we say that K is a *proper π–domain.*

By $\mathbf{MOD}_\sigma$ we mean the class of all finite models of vocabulary σ. Because we consider only finite vocabularies then there is only countably many non isomorphic models in $\mathbf{MOD}_\sigma$. Moreover we can assume that all finite models are defined on the initial segments of natural numbers.[6] Therefore we can assume that $\mathbf{MOD}_\sigma$ is a reasonable, computationally manageable, countable set.

[6] Of course it does not mean that we consider only arithmetical models. We use only the fact that each finite model is isomorphic with a model having the set $\{0, 1, \ldots, n-1\}$ as the universe, for some n.

In what follows we will frequently use the set of purely logical sentences $A_{inf} = \{\xi_1, \xi_2, \xi_3, \ldots\}$, where ξ_n is the following:

$$\forall x_1 \ldots \forall x_n \exists y (x_1 \neq y \wedge \ldots \wedge x_n \neq y).$$

The sentence ξ_n says that *there are more than n objects.*

4.2 First order logic in sl–semantics

When a vocabulary σ is fixed then sl–logic L_{sl} we define as $L_{sl} = sl(\mathbf{MOD}_\sigma)$. The logic of finite models L_{fin} is defined as $L_{fin} = th(\mathbf{MOD}_\sigma)$, where

$$th(K) = \{\varphi \in F_\sigma : M \models \varphi, \text{ for all } M \in K\}.$$

Of course we have $L_{fin} \subseteq L_{sl}$. Additionally the inclusion is proper because each statement ξ_n, for $n > 0$, belongs to $L_{sl} - L_{fin}$, where ξ_n is the following:

$$\forall x_1 \ldots \forall x_n \exists y (x_1 \neq y \wedge \ldots \wedge x_n \neq y).$$

The sentence ξ_n says that *there are more than n objects.*

Let us define the set A_{sl} as the union of L_{fin} and $\{\xi_1, \xi_2, \ldots\}$, so $A_{sl} = L_{fin} \cup A_{inf}$.

Theorem 4.1 *L_{sl} is the set of all first oder consequences of A_{sl}.*

4.3 Complexity of sl–logic

The main advantage of axiomatic method is the possibility of giving conclusive arguments for truth of our claims in a way that soundness of these arguments can be checked in a routine way. Therefore sets of theorems of practically useful theories have to be recursively enumerable or equivalently Σ^0_1. This is so because φ *is a theorem* is equivalent to:

$$\exists D \ (D \textit{ is a proof for } \varphi),$$

and the relation "*is a proof for*" should be recursive.

The existence of complete proof procedure is the main advantage of first order logic in comparison with stronger logical systems. Therefore applicability of first order logic for theories based on sl–semantics seems to be a good argument for plausibility of sl–semantics. Unfortunately things are not so simple.

We start with recalling a classical result.

Theorem 4.2 (Trachtenbrot's theorem) *The set L_{fin} is Π^0_1–complete.*

As a corollary we obtain the following.

Theorem 4.3 *The set A_{sl} is Π^0_1–complete.*

On the other hand we have the following:

Theorem 4.4 *The set L_{sl} is Σ^0_2–complete.*

4.4 The completeness of first order logic with respect to sl–semantics

The classical completeness theorem[7] says that for each $T \subseteq F_\sigma$ and $\varphi \in F_\sigma$ we have the equivalence

$$T \vdash \varphi \text{ if and only if } T \models \varphi,$$

where on the left side we have the standard provability relation $\vdash$, and on the right side we have the standard semantical entailment $\models$. What will happen if we replace $\models$ by $\models_{sl}$?

The answer is given by the following:

Theorem 4.5 (The Completeness Theorem) *For each $T \subseteq F_\sigma$ containing A_{sl} ($A_{sl} \subseteq T$) and $\varphi \in F_\sigma$ we have the equivalence*

$$T \vdash \varphi \textit{ if and only if } T \models_{sl} \varphi,$$

and the same holds when we restrict semantics to proper π–domains.

5 A few final remarks

Studying mathematics without actual infinity – as a logical topic – is rather new research field. There is a lot of philosophical discussions of particular relevant questions and very few of mathematical investigations in this direction.

Summarizing our short survey of ideas and results we list a few general observations:

[7]The completeness theorem was proved for the first time in earlier mentioned paper by Kurt Gödel. However in this general form it was proved by Leon Henkin [5].

- Rejecting actual infinity we can represent infinity by potentially infinite domains which determine possible ways of getting infinity.
- These ways of getting infinity can be described in terms of games played by recursive players whose moves larger finite interpretations – approximating infinity – which support their claims.
- Truth without potential infinity is much simpler than that of actual infinity. Nevertheless, it is slightly more complicated than theoremhood in a given axiomatic framework.
- Finally, a historical remark. Mathematics without actual infinity was dominating mathematical thinking until the half of twentieth century, and it never disappeared.

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