COMPUTATION IN AND ABOUT PHYSICS AND MATHEMATICS

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Abstract. In Mathematics, the statement *There exists an object x such that $A(x)$* means that there is an object that satisfies the formula A, usually without much information as to how to construct or compute such an object, if that is possible at all. By contrast, in Physics, existence statements such as the above are invariably accompanied by a computation or construction of the object at hand (either symbolically or numerically). Naturally, the question arises why the Mathematics in Physics is so *computable and/or constructive*? In this talk, we argue that the combination of the mathematical practice in Physics (an intuitive calculus with infinitesimals), together with its very nature (the modeling of real-world phenomena), always gives rise to computable and/or constructive objects. The key to this answer is Ω*-invariance*: a new and simple definition of the fundamental notions *algorithm* and *finite procedure* inside Nonstandard Analysis.

1. Main Question

In this paper, we answer the following question.

Question. *Why is the Mathematics in Physics so constructive and/or computable?*

To this end, we first discuss the meaning of the latter technical terms. Secondly, we discuss the nature of Mathematics in Physics, and answer the above question.

2. The origins of computability and constructivity

In 1928, the mathematician David Hilbert posed the *Entscheidungsproblem*. In modern language, this problem asks for the construction of an algorithm deciding the truth or falsity of a mathematical statement (Davis, 1965). Before the Entscheidungsproblem could be solved, a formal definition of algorithm was necessary and Alan Turing provided such a formalism (Turing, 1937), which later became known as the 'Turing Machine'. Intuitively, the latter corresponds to a modern-day computer with no limitation on memory or running time. Although Turing's formalism provides a negative answer to Hilberts question, as did Alonzo Church's *lambda calculus* (Church, 1936a; 1936b), a positive outcome of Church and Turing's work is the formalization of the -until then vague- class of 'effectively computable' functions as the functions computable on a Turing machine. The latter assertion is called the *Church-Turing thesis* (Davis, 1965, p. 100-102, 274).

The notion *constructive* also has historical ties to David Hilbert. Hilbert pioneered the first proofs by contradiction in Mathematics (Ewald, 1996). Such proofs only imply the *existence* of an object, but in general there is no way to generate this object, as was customary until then in Mathematics. For L.E.J. Brouwer, such proofs were unacceptable, as he believed that objects only exists after they

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have been constructed (van Heijenoort, 1967). Brouwer formulated an alternative Mathematics, and accompanying philosophy, centered around the rejection of the tertium non datur (principle of excluded middle), and based on intuition rather than formalism. Thus, intuitionism was born at the turn of the 19th century (van Heijenoort, 1967). Arend Heyting provided a formalization of intuitionistic logic (van Heijenoort, 1967) and Errett Bishop based his famous *Constructive Analysis* (Bishop, 1967) on the system developed by Brouwer, Heyting and others. In each case, an object only exists after it has been constructed. It is in this sense that we use the notion 'construction'. It is closely related to, but distinct from, the notion 'computable on a Turing Machine'.

3. The nature of Mathematics in Physics

With regard to the nature of Mathematics in Physics, historically, an intuitive and informal 'calculus with infinitesimals' was used in Mathematics. For various reasons, this framework was abandoned in favor of Karl Weierstraß' notorious ε-δ-method. In Physics however, Weierstraß' framework was never adopted and an informal calculus with infinitesimals is used to date. Furthermore, the level of mathematical rigor is much lower in physics than in mathematics, for various reasons (Davey, 2003). An often heard rule of thumb in physics is that *As long as the end result has physical meaning, the mathematical manipulation to get there need not be completely rigorous.*

We now consider an essential independence property particular to Physics. Suppose x is the physically meaningful end result of a mathematical manipulation M in Physics, involving infinitesimals. As infinitesimals are generally believed to be mere calculus tools (without real-world existence), the end result x should be independent of the *choice* of infinitesimals in M. Indeed, the choice of calculus tool (in casu the infinitesimals used in M) should have no effect on the end result x. In short, we may assume that an end result in Physic does *not* depend on the *choice* of infinitesimals used, as the latter are merely calculus tools.

4. Answering the main question

4.1. Introducing Nonstandard Analysis and Ω-invariance.

Around 1960, Abraham Robinson introduced *Nonstandard Analysis*, providing a formalization of the calculus with infinitesimals (Robinson, 1966). In Nonstandard Analysis, the set $\mathbb{N}$ is extended with new elements which are larger than all $n \in \mathbb{N}$. The resulting set is called ${}^*\mathbb{N}$ and any $\omega \in \Omega := {}^*\mathbb{N} \setminus \mathbb{N}$ is called *infinite*. In contrast, any $n \in \mathbb{N}$ is called *finite*. When working with $\mathbb{Q}$ or $\mathbb{R}$ instead of $\mathbb{N}$, the inverse of an infinite number is called an *infinitesimal*.

The following definition captures the idea, discussed in the previous paragraph, that a set A is independent of the choice of infinitesimal in its definition.

1. **Definition.** [Ω-invariance] A set $A \subset \mathbb{N}$ is Ω-invariant, if there exists a quantifier-free formula $\psi(n, m)$ such that $A = \{n \in \mathbb{N} : \psi(n, \omega)\}$ for *all* $\omega \in \Omega$.

As it turns out, the notion of Ω-invariance captures the notions of computation and construction quite well. We briefly sketch some results from (Sanders, 2011; 2013; 2012) that support this claim.

First of all, the following theorem states that the properties of an Ω-invariant set are already determined at some *finite* number. This observation suggests that Ω-invariance models the notion of *finite procedure* quite well.

2. **Theorem** (Modulus lemma). *For every Ω-invariant set $A = \{n \in \mathbb{N} : \psi(n, \omega)\}$,*

$$(\forall n \in \mathbb{N})(\exists m_0 \in \mathbb{N})(\forall m \in {}^*\mathbb{N})[m \geq m_0 \to \psi(n, m) \leftrightarrow n \in A],$$

and the number m_0 can be computed by an Ω-invariant function.

The previous theorem is called 'modulus lemma' as it bears a resemblance to the modulus lemma from Recursion Theory (Soare, 1987, 3.2).

4.2. **Turing Computability and Ω-invariance.** Here, we consider the well-known *Limit Lemma* for limit computable functions from (Soare, 1987, 3.3).

3. **Theorem** (Limit Lemma). *For any function f,*

$$f \leq_T \mathbf{0}' \leftrightarrow f \in \Delta_2 \leftrightarrow f = \lim_{n \to \infty} f_n \qquad (f_n \text{ is computable})$$

Here, '$f \leq_T \mathbf{0}'$' means that f can be computed by a Turing machine with a finite number of queries to the oracle $\mathbf{0}'$. The latter is called the *Turing jump* and provides a decision procedure for existential formulas.

In Nonstandard Analysis, we have the following theorem (Sanders, 2011).

4. **Theorem** (Hyperlimit Lemma). *For any function f*

$$f \leq_\Omega \mathbf{\Pi}' \leftrightarrow f \in \Delta_2 \leftrightarrow f = f_\omega \qquad (f_n \textit{ is computable, } \omega \in \Omega)$$

Note that the function f_ω on the right-hand side is Ω-invariant. Furthermore, we write '$f \leq_\Omega \Pi'$' to denote that there is an Ω-invariant procedure to compute f, with $\mathbf{\Pi}'$ as an oracle. The latter is an Ω-invariant decision procedure for existential formulas, using *Universal Transfer* as an oracle.

5. **Principle** (Universal Transfer). *For all q.f. φ, $(\forall n \in \mathbb{N})\varphi(n) \to (\forall n \in {}^*\mathbb{N})\varphi(n)$.*

Transfer is a central principle from Nonstandard Analysis (Robinson, 1966). Thus, it seems that Ω-invariance captures the notion of Turing machine (and associated notions) quite well.

4.3. **Constructive Analysis and Ω-invariance.** In *Constructive Analysis*[1], the notion of (constructive) *algorithm* is central. This already becomes clear from the definition of disjunction (Bridges, 1999; Bishop, 1967).

6. **Definition.** [Disjunction] $P \vee Q$: we have an algorithm that outputs either P or Q, together with a proof of the chosen disjunct.

In Nonstandard Analysis, we have the following similar definition (Sanders, 2012), where the role of 'algorithm' is played by Ω-invariance.

7. **Definition.** [Hyperdisjunction] For formulas P, Q the formula $P \veebar Q$ is the statement: *There is an Ω-invariant formula ψ such that*

$$(\forall n \in \mathbb{N})(\psi(n, \omega) \to P(n) \wedge \neg\psi(n, \omega) \to Q(n)).$$

Given the formula $P \veebar Q$, there is an Ω-invariant procedure, provided by $\psi(n, \omega)$, to determine which disjunct of $P(n) \vee Q(n)$ makes it true. Thus, we observe that the meaning of the hyperdisjunction '$\veebar$' is quite close to its intuitionistic counterpart '$\vee$' from Definition 6.

The other intuitionistic connectives may be translated analogously. The translation of $\to$ (resp. $\neg$) will be denoted $\Rrightarrow$ (resp. $\sim$). As for disjunction, the meaning of the intuitionistic connectives is quite close to that of the hyperconnectives.

[1]Constructive analysis is also called 'BISH', in honour of its founder, Erret Bishop

These new connectives provide a translation of BISH into Nonstandard Analysis. As it turns out, this translation preserves the many equivalences from *Constructive Reverse Mathematics*[2] (Ishihara, 2006). For instance, compare the following theorems.

8. **Theorem.** *In* BISH*, the following are equivalent.*

(1) LLPO: $\neg(P \wedge Q) \to \neg P \vee \neg Q \quad (P, Q \in \Sigma_1)$.
(2) LLPR: $(\forall x \in \mathbb{R})[\neg(x > 0) \vee \neg(x < 0)]$.
(3) NIL: $(\forall x, y \in \mathbb{R})(xy = 0 \to x = 0 \vee y = 0)$.
(4) CLO: *For all* $x, y \in \mathbb{R}$ *with* $\neg(x < y)$, $\{x, y\}$ *is a closed set.*
(5) IVT: *a version of the intermediate value theorem.*
(6) WEI: *a version of the Weierstraß extremum theorem.*

9. **Theorem.** *In Nonstandard Analysis, the following are equivalent.*

(1) $\mathbb{LLPO}$: $\sim(P \wedge Q) \Rrightarrow \sim P \Vvee \sim Q \quad (P, Q \in \Sigma_1)$.
(2) $\mathbb{LLPR}$: $(\forall x \in \mathbb{R})[\sim(x > 0) \Vvee \sim(x < 0)]$.
(3) $\mathbb{NIL}$: $(\forall x, y \in \mathbb{R})(xy = 0 \Rrightarrow x = 0 \Vvee y = 0)$.
(4) $\mathbb{CLO}$: *For all* $x, y \in \mathbb{R}$ *with* $\sim(x < y)$, $\{x, y\}$ *is a closed set.*
(5) $\mathbb{IVT}$: *a version of the intermediate value theorem.*
(6) $\mathbb{WEI}$: *a version of the Weierstraß extremum theorem.*

In (Sanders, 2012), Sanders shows that most equivalences from Constructive Reverse Mathematics (e.g. those for LPO, LLPO, WLPO, MP, $MP^{\vee}$, FAN_{Δ} and WMP) can be translated to Nonstandard Analysis. Hence, we observe that Ω-invariance must be close to Bishop's notion of (constructive) algorithm, as it gives rise to exactly the same kind of Reverse Mathematics results.

4.4. **Conclusion.** In light of the results in Sections 4.2 and 4.3, we conclude that the notion of Ω-invariance is quite close to both the notions of computation and construction. These results give rise to the following answer to our main question.

Answer. The Mathematics in Physics usually takes place in an informal 'calculus with infinitesimals' where the end result should have physical meaning. The latter assumption in particular implies that end results are independent of the *choice* of infinitesimals used in this calculus. By recent results, this independence condition, formally called Ω-invariance, captures the notions of construction and computation quite well. In particular, this correspondence between the latter and Ω-invariance explains why Mathematics in Physics is computable and/or constructive.

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[2]See (Simpson, 2009) for an introduction to Reverse Mathematics.

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