From Computability to Information Theory: Effective fractal dimension in general spaces

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Universidad de Zaragoza

11 October 2012

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Turing and normality

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- He then uses this proof to effectively construct an absolutely normal number
- Today these are “classic ideas” but they did not emerge until 20 years later

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- We now know that normality is a type of randomness


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Algorithmic randomness

- Can we generate randomness?
- Can we quantify randomness?
- What can we compute using randomness?

Opinions ...

- Turing 1950:

  An interesting variant on the idea of a digital computer is a "digital computer with a random element". These have instructions involving the throwing of a die or some equivalent electronic process; one such instruction might for instance be, "Throw the die and put the-resulting number into store 1000". Sometimes such a machine is described as having free will (though I would not use this phrase myself).

- von Neumann 1951:

  Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin.

Definitions of algorithmic randomness

Rod Downey, plenary at CiE 2012:

- the measure theory approach: Abundance.
- the gambler's approach: Unpredictability.
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Today I will be mostly interested in **infinite** random objects (infinite binary/nonbinary sequences, real numbers, ...)

Three ways to calibrate and extend algorithmic randomness

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Three ways to calibrate and extend algorithmic randomness

- Resource bounds: Consider complexity bounds on the generation, quantification and use of randomness
- Partial randomness: Consider objects that contain some amount of randomness even if not maximal
- General spaces: Consider spaces more general than the usual Cantor space

Resource-bounds on randomness

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- Very well studied, natural for computability approaches

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- A lot known on the interaction of randomness and Turing degrees
- A bit oversized for Computational Complexity, still it can be used to define classes such as BPP

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- It inherits non measurability issues from Martin-Löf approach

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- (New hypothesis) If NP contains p-random sets then Turing and many-one polynomial time completeness differ in NP.

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- Can be used with most resource-bounds

A taste of effective dimension

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- Zero-One laws: dim(BPP|EXP)$= 0$ or BPP=EXP
- Back to normality: Finite-State randomness coincides with Finite-State dimension $= 1$.

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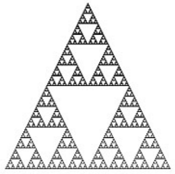
- Hausdorff dimension is well studied over Euclidean space, effective dimension has meaningful geometric results too
- Effective dimension in other spaces would give a (partial) randomness concept in those (e.g. dynamical systems)

In Euclidean space

Effective dimension in Euclidean space has analyzed the dimension of points in

- self-similar fractals,
- random self-similar fractals,



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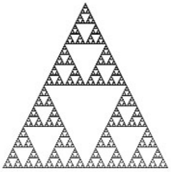

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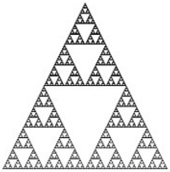

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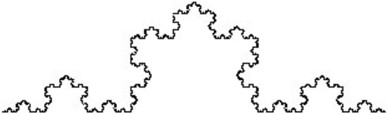
Hausdorff, 1919: Rigorous formulation of dimension.



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$H^s(X) =$ the s-dimensional Hausdorff measure of X

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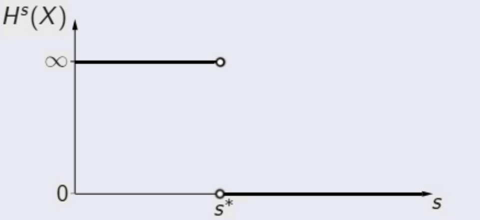
Definition (Fractal Dimension)

Let ρ be a metric on $\mathcal{X}$, and let $X \subseteq \mathcal{X}$.

- (Hausdorff 1919) The Hausdorff dimension of X is

$$\dim_{\mathrm{H}}(X) = \inf \{ s \mid H^s(X) = 0 \} .$$



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3. (c-cover) For every $r \in \mathbb{N}$ there is an $\epsilon > 0$ such that for every $A \subseteq \mathcal{X}$ with $0 < \mathrm{diam}(A) < \epsilon$ there exists $\{U_1, \ldots, U_c\} \subseteq \cup_{n>r}\mathcal{B}_n$ a cover of A, with $\mathrm{diam}(U_i) < c \cdot \mathrm{diam}(A)$ for every i.

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c is a constant for the whole space

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  $\mathcal{C}_w = \{x \,|\, w \sqsubseteq x, x \in \{0,1\}^\infty\}$

Given a nice collection of covers of $\mathcal{X}$

- You can approximate each $x \in \mathcal{X}$, with $(x \uparrow n)$ such that
  - $x \in x \uparrow n$
  - $x \uparrow n \in \mathcal{B}_n$

Gales on $\mathcal{X}$

$\mathcal{B} = \cup_n \mathcal{B}_n$

- An s-gale is a function $d : \mathcal{B} \to [0, \infty)$ such that for $U \in \mathcal{B}_n$

$$d(U)\,\mathrm{diam}(U)^s = \sum_{V \subseteq U, V \in \mathcal{B}_{n+1}} d(V)\,\mathrm{diam}(V)^s$$

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Martingales are strategies for betting on the successive approximations of $x \in \mathcal{X}$, and one of these strategies succeeds on x if it makes an infinite amount of money betting on x.

Gales are generalized martingales that are no more powerful, but exhibit the martingales' success rates in a convenient form.

In Cantor space

For $\mathcal{X} = \{0,1\}^\infty$, with $\mathcal{B}_n = \{\mathcal{C}_w \,|w \in \{0,1\}^n\}$, an s-gale is $d : \{0,1\}^* \to [0,\infty)$ with

$$d(w) = \frac{d(w0) + d(w1)}{2^s}$$

Characterization

An s-gale is a function $d : \mathcal{B} \to [0, \infty)$ such that for $U \in \mathcal{B}_n$

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The success set of an s-gale is

$$S^\infty[d] = \left\{ x \in \mathcal{X} \,\middle|\, \limsup_n d(x \upharpoonright n) = \infty \right\}$$

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$\mathcal{G}(A) = \{ s \mid$ there is an s-gale d such that $A \subseteq S^\infty[d] \}$

A gambling characterization of dimension

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Theorem

For every $A \subseteq \mathcal{X}$, $\dim_{\mathrm{H}}(A) = \inf \mathcal{G}(A)$.

Effectivizing dimension

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An s-gale is a function $d : \mathcal{B} \to [0, \infty)$ such that for $U \in \mathcal{B}_n$

$$d(U)\,\mathrm{diam}(U)^s = \sum_{V \subseteq U, V \in \mathcal{B}_{n+1}} d(V)\,\mathrm{diam}(V)^s$$

$$\dim_{\mathrm{H}}(A) = \inf\{s \mid \text{there is an } s\text{-gale } d \text{ such that } A \subseteq S^{\infty}[d]\}$$

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We can define effective versions by restricting to classes of computable gales.

Constructive Dimension

Let $\mathcal{X}$ be a metric space with a nice computable collection of covers, that is

- (Computable diameter) $\mathcal{B} = \cup_n \mathcal{B}_n$ is countable and there is a surjective $\delta : \Sigma^* \to \mathcal{B}$ for a finite Σ such that $\mathrm{diam} \circ \delta$ is computable

Constructive Dimension

Definition

An s-gale d is constructive if $d \circ \delta$ is lower semi-computable, i.e., if there is an exactly computable function $\hat{d} : \Sigma^* \times \mathbb{N} \to \mathbb{Q}$ with the following two properties.

- For all $w \in \Sigma^*$ and $t \in \mathbb{N}$, $\hat{d}(w,t) \leq \hat{d}(w,t+1) < d(w)$.
- For all $w \in \Sigma^*$, $\lim_{t\to\infty} \hat{d}(w,t) = d \circ \delta(w)$.

Constructive Dimension

Definition

Let $A \subseteq \mathcal{X}$.

- The constructive dimension of A is

$$\mathrm{cdim}(A) = \inf \left\{ s \;\middle|\; \begin{array}{l} \text{there is a constructive } s\text{-gale } d \\ \text{such that } A \subseteq S^{\infty}[d] \end{array} \right\}$$

Characteristics of constructive dimension

- It is non necessarily zero and meaningful on singletons

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- It coincides with Hausdorff dimension in many interesting cases
- It can be characterized in terms of Kolmogorov complexity

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Absolute Stability of Constructive Dimension

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For all $A \subseteq \mathcal{X}$,

$$\mathrm{cdim}(A) = \sup_{x \in A} \dim(x).$$

(Contrast with countable stability of classical dimension.)

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$\therefore$ Constructive dimension is investigated in terms of individual points.

Correspondence principle of constructive dimension

A correspondence principle for an effective dimension is a theorem stating that, on sufficiently simple sets, the effective dimension coincides with its classical counterpart. (Terminology stolen from N. Bohr by Lutz.)

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Correspondence Principle for Constructive Dimension

Theorem

If $A \subseteq \mathcal{X}$ is any union (not necessarily effective) of computably closed (i.e., Π^0_1) sets then $\mathrm{cdim}(A) = \dim_{\mathrm{H}}(A)$.

Kolmogorov complexity characterization

Definition

Let $x \in X$, let $r \in \mathbb{N}$. The Kolmogorov complexity of x at precision r is

$$\mathrm{K}_r(x) = \inf\left\{\mathrm{K}(w) \,\middle|\, x \in \delta(w),\, 2^{-r} < \mathrm{diam}(\delta(w)) \leq 2^{-r+1}\right\},$$

with $\mathrm{K}_r(x) = \infty$ if not such w exists.

Theorem

Let X be a metric space with a computable nice cover. Let $x \in X$,

$$\mathrm{cdim}(x) = \liminf_r \frac{\mathrm{K}_r(x)}{r}.$$

Ongoing work

- Constructive dimension as a source of randomness in spaces other than Cantor and Euclidean
- In particular those related to dynamical systems

Conclusions

- Measure effectivization, prediction, and Kolmogorov complexity give robust randomness concepts
- Resource-bounds and effective dimension have been used to calibrate and generalize Martin-Löf randomness
- Effective dimension can be generalized to other metric spaces using the gambling characterization of dimension

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