First-Order and Existential Definability and Decidability in Positive Characteristic

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October 2012

Outline

Hilbert's Question about Polynomial Equations

Is there an algorithm which can determine whether or not an arbitrary polynomial equation in several variables has solutions in integers?

This problem became known as Hilbert's Tenth Problem

The Answer

This question was answered negatively (with the final piece in place in 1970) in the work of Martin Davis, Hilary Putnam, Julia Robinson and Yuri Matijasevich.

A General Question

A Question about an Arbitrary Recursive Ring R

Is there an algorithm, which if given an arbitrary polynomial equation in several variables with coefficients in R, can determine whether this equation has solutions in R?

Arguably, the most important open problems in the area concern the Diophantine status of the ring of integers of an arbitrary number field and the Diophantine status of $\mathbb{Q}$.

Does Hilbert's Question Make Sense over Uncountable Rings?

Yes, it does make sense to consider uncountable rings

as long as we consider polynomial equations with coefficients restricted to a countable recursive subring. We can still consider solutions in the bigger ring. In other words, given a polynomial equation with coefficients in a fixed finitely generated ring, we will consider existence of an algorithm which can take the coefficients as inputs and determine whether solutions exist in the bigger, possibly uncountable ring.

Outline

Semigroups and Groups

Definition (Semigroup)

Given a set G, a binary operation on G is a map (a function) from ordered pairs of elements of G into G. An operation $\cdot$ is commutative if for all elements $a, b \in G$ we have that $a \cdot b = b \cdot a$. A semigroup is a set together with a binary operation $\cdot$ that satisfies the associative property: for all elements a, b, c of the set we have that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Definition (Group)

A semigroup G is a group if

1. there exists an element e, called the *identity*, such that for any element $a \in G$ we have that $a \cdot e = e \cdot a = a$, and
2. for every element $a \in G$ there exists a $b \in G$ with $a \cdot b = b \cdot a = e$ (every element has an *inverse*).

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Rings

Definition (Ring)

Given a set R with two binary operations $+$ ("addition") and $\cdot$ ("multiplication"), we say that R is a ring if

1. R is a group under addition with the identity element 0, and addition is commutative,
2. R is a semigroup under multiplication with the identity element 1, and
3. for all elements $a, b, c \in R$ we have that $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$, i.e. distributive law holds.

If multiplication is commutative, then R is called a commutative ring.

Definition (Ideal of a Ring)

Let R be a ring and let I be a subset of the ring such that it is a group under the ring addition and it is closed under multiplication by the elements of the ring, i.e. for every $x \in I$ and every $a \in R$, we have that $ax \in I$. In this case we say that I is an ideal of R. If for any $x, y \in R$ we have that $xy \in I$ implies either $x \in I$ or $y \in I$, then we say that the ideal is prime.

Definition (Power of an Ideal)

Let R be a ring and let I be an ideal of R. In this case, for any positive integer n we define I^n to be the set of all elements of R that can be written in the form $\sum_{i=1}^{k} a_{i,1} \dots a_{i,n}$ for some positive integer k, where all $a_{i,j} \in I$.

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Fields

Definition (Field)

A commutative ring R is a field if $R \setminus \{0\}$ is a group under multiplication, i.e. every non-zero element has a multiplicative inverse.

Outline

Fields of Positive Characteristic

Definition

Let p be a prime number and let k be a field such that for any element x of the field $px = 0$. (Here by px we mean x added to itself p-times.) In this case we say that the field has characterisitic p.

Example

For any prime number p it is the case that $\mathbb{Z}/p$ is a field of characteristic p with p elements. Any field of characteristic p contains $\mathbb{Z}/p$ as a subfield.

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An example of a finite field: $\mathbb{Z}/3$

$\mathbb{Z}/3$ can be thought of as a field of remainders with respect to division by 3. It has three elements which we can denote by $\{0, 1, 2\}$. Using this notation, each time we perform a field operation we should replace the result of "normal" addition or multiplication by its remainder from division by 3. For example in $\mathbb{Z}/3$ we have that $2 + 1 = 0$ and $2 \cdot 2 = 1$.

Polynomial Rings and Fields of Rational Functions Over Finite Fields

Polynomial Rings over a Finite Field

A polynomial ring over a finite field is a set of polynomials in one or more variables. For example a polynomial ring in one variable x over a finite field k, denoted by $k[x]$, is the set of all elements of the form $a_0 + a_1x + \ldots + a_nx^n$, where n is a non-negative integer and $a_0, \ldots, a_n \in k$.

A Field of Rational Functions over a Finite Field

If we consider the set of all ratios of polynomials with coefficients in a finite field k, we obtain a field of rational functions over a finite field. Such a field in one variable x, denoted by $k(x)$, contains all ratios of the form $\frac{a_0+a_1x+\ldots+a_nx^n}{b_0+b_1x+\ldots+b_mx^m}$, where m, n are non-negative integers, $a_0, \ldots, a_n, b_0, \ldots, b_m \in k$, not all $b_0, \ldots, b_m$ are equal to zero, and two ratios represent the same field element if they are equal under cross-multiplication.

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Transcendental Elements

Note that in the field $k(x)$ it is the case that x is not a solution of any non-trivial polynomial equation with coefficients in k. In other words, if $a_0 + a_1x + \ldots + a_nx^n = 0$, then $a_0 = a_1 = \ldots = a_n = 0$. We say that x is transcendental over k.

Field Extensions

If $F \subseteq G$ are fields, then we say that G is a field extension of F and write G/F. If all the elements of G are roots of non-zero polynomials with coefficients in F, we say that the extension is algebraic. Further, if all the elements of G satisfy non-zero polynomial equations with coefficient in F of degree less or equal to a positive integer n, we say that the extension is finite. Further, if n is the smallest positive integer that "works", we say that the degree of the extension is n and write $[G : F] = n$.

Algebraic Closure

A field is algebraically closed if every polynomial with coefficients in the field has a root in the field. Given a field k, its algebraic closure is the smallest algebraically closed field containing k.

Transcendence Degree of a Field

Definition (Algebraic Independence)

Let K/k be a field extension, and let $y_1, \ldots, y_n \in K$. In this case, if for any polynomial $P(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n]$ we have that $P(y_1, \ldots, y_n) = 0 \iff P(X_1, \ldots, X_n) \equiv 0$ as an element of $k[X_1, \ldots, X_n]$, and $n \in \mathbb{Z}_{\geq 2}$, we say that $y_1, \ldots, y_n$ are algebraically independent over k. If $n = 1$, then we say that y_1 is transcendental over k.

Definition (Transcendence Degree)

If K/k is a field extension, then the transcendence degree of K/k is the size of the largest subset of elements of K algebraically independent over k.

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Algebraic Function Fields of Positive Characteristic

Definition (Algebraic Function Field of Positive Characteristic)

Let k be a field of positive characteristic, let $k(t)$ be a rational function field over k in one variable, and let $K/k(t)$ be a (algebraic) finite extension. In this case K is a (algebraic) function field of positive characteristic in one variable. The field of constants of K is the algebraic closure of k in K, i.e. the largest algebraic extension of k inside K.

Transcendence Degree of Function Fields

Remark

A function field in one variable is of transcendence degree 1 over its field of constants. One can also consider the transcendence degree of a function field over $\mathbb{Z}/p$. This degree can be any positive integer or infinity.

Outline

HTP over Rational Function Fields of Positive Characteristic

Theorem

HTP is unsolvable over the following fields:

- *rational function fields over finite fields of characteristic greater than 2 (Pheidas, 1991);*
- *rational function fields over a constant field k, where k is a proper subfield of the algebraic closure of a finite field (Kim and Roush, 1992).*
- *rational function field of a finite transcendence degree greater or equal to two over the algebraic closure of a finite field of odd characteristic (Kim and Roush, 1992).*
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HTP over Algebraic Function Fields of Positive Characteristic of Transcendence Degree 1

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HTP is unsolvable over the following fields:

- *algebraic function fields over finite fields of characteristic greater than 2 (S. 1996);*
- *algebraic function fields over fields of constants k algebraic over $\mathbb{Z}/p$ and having an extension of degree $p > 2$ (S. 2000);*
- *fields as above for $p = 2$ (Eisenträger 2003)*

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HTP over Algebraic Function Fields of Positive Characteristic of Higher Transcendence Degree

Theorem

HTP is unsolvable over the following fields.

- *a field $K = k(u, v) \otimes_{\mathbb{Z}/p} F$, where $p > 2$, k is algebraic over $\mathbb{Z}/p$ and has an extension of degree p, u is transcendental over k, v is algebraic over $k(u)$, and $k(u, v)$ and F linearly disjoint over $\mathbb{Z}/p$ (S. 2000);*
- *K as above for $p = 2$ (Eisenträger 2003)*
- *any field K finitely generated over $\mathbb{Z}/p$ (S. 2002)*
- *a field $K = E \otimes_{\mathbb{Z}/p} F$, where E is finitely generated over a field k algebraic over $\mathbb{Z}/p$ and with an extension of degree p, and E and F are linearly disjoint over $\mathbb{Z}/p$ (S. 2003)*
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Outline

Completing the Extension of Kim and Rousch

Theorem (K. Eisentraeger and S, work in progress)

Let K be any function field of positive characteristic not containing the algebraic closure of a finite field. In this case HTP is undecidable over K.

Completing the Proof of the First-Order Undecidability

Theorem (Eisentraeger, S. , work in progress)

If K is any function field of positive characteristic, then the first-order theory of K in the language of rings is undecidable.

The Main Unsolved Question

A Problem

Let C_p be the algebraic closure of $\mathbb{Z}/p$ for some rational prime p. Show that HTP over a function field (or even a rational function field) over C_p is undecidable.

Outline

Function Fields

Definition (Algebraic and Integral Functions)

Algebraic functions are roots of polynomials with coefficients in a field of rational functions, in our case $k(t)$. If γ is an algebraic function, then it is an integral function if it satisfies a *monic* irreducible over $k(t)$ polynomial with coefficients in the polynomial ring $k[t]$.

Example

$\sqrt{t^2+1}$ is a root of a monic irreducible polynomial $X^2-(t^2+1)=0$. Thus, $\sqrt{t^2+1}$ is an integral function. At the same time $\sqrt{\frac{t+1}{t-1}}$ is a root of the polynomial $(t+1)X^2-(t-1)$, which is irreducible over $\mathbb{Z}/p(t)$ and has polynomial coefficients but is not monic. To make $(t+1)X^2-(t-1)$ monic we have to allow rational function coefficients, and therefore $\sqrt{\frac{t+1}{t-1}}$ is not an integral function.

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Integral Functions and Primes of Function Fields

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If K is a function field over a field of constants k (and a finite extension of $k(t)$), then the set of all functions integral over $k[t]$ form a ring O_K which we call the ring of integral functions of K. We will also consider the integral closure in K of $k[\frac{1}{t}]$ and denote that ring by $O_{K,\infty}$.

Definition

A prime of a function field K is a prime ideal of O_K or a prime ideal of $O_{K,\infty}$. The prime ideals of $O_{K,\infty}$ are referred to as *infinite primes*.

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Order at a Prime over Global Function Fields

Order at a Prime from O_K over a Function Field

If K is a global function field, $x \neq 0$ and $x \in O_K$, then for any prime $\mathfrak{p}$ of K originating in O_K there exists a non-negative integer m such that $x \in \mathfrak{p}^m$ but $x \notin \mathfrak{p}^{m+1}$. We call m the order of x at $\mathfrak{p}$ and write $m = \operatorname{ord}_{\mathfrak{p}} x$. If $y \in K$ and $y \neq 0$, we write $y = \frac{x_1}{x_2}$, where $x_1, x_2 \in O_K$ with $x_1 x_2 \neq 0$, and define $\operatorname{ord}_{\mathfrak{p}} y = \operatorname{ord}_{\mathfrak{p}} x_1 - \operatorname{ord}_{\mathfrak{p}} x_2$. This definition is not dependent on the choice of x_1 and x_2 which are of course not unique. We define $\operatorname{ord}_{\mathfrak{p}} 0 = \infty$ for any prime $\mathfrak{p}$ of O_K.

Order at a Prime from $O_{K,\infty}$ over a Function Field

The order at the primes which are ideals of $O_{K,\infty}$ are defined in the analogous manner with $O_{K,\infty}$ substituting for O_K.

Order at a Prime over Global Function Fields

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Primes of a Rational Function Field

In the case $K = k(t)$ all but one prime correspond to irreducible polynomials in t and the remaining (infinite) prime corresponds to the degree of polynomials. For example, consider $x = \frac{t^2+1}{t-1}$ over k where -1 is not a square. Let $\mathfrak{p}_1$ correspond $t^2 + 1$, $\mathfrak{p}_2$ correspond to $t - 1$, $\mathfrak{p}_\infty$ correspond to degree. In this case,

$$\operatorname{ord}_{\mathfrak{p}_1} x = 1,$$

$$\operatorname{ord}_{\mathfrak{p}_2} x = -1,$$

$$\operatorname{ord}_{\mathfrak{p}_\infty} x = \operatorname{ord}_{\mathfrak{p}_\infty}(t^2 + 1) - \operatorname{ord}_{\mathfrak{p}_\infty}(t - 1) = -2 - (-1) = -1.$$

Properties of Order

If $x, y \in K$, and $\mathfrak{q}$ is a prime of K, then

$$\operatorname{ord}_{\mathfrak{q}}(xy) = \operatorname{ord}_{\mathfrak{q}}(x) + \operatorname{ord}_{\mathfrak{q}}(y).$$

In particular,

$$\operatorname{ord}_{\mathfrak{q}}(x^r) = r \operatorname{ord}_{\mathfrak{q}}(x).$$

Further, $\operatorname{ord}_{\mathfrak{q}}(x+y) \geq \min(\operatorname{ord}_{\mathfrak{q}} x, \operatorname{ord}_{\mathfrak{q}} y)$ and if $\operatorname{ord}_{\mathfrak{q}} x < \operatorname{ord}_{\mathfrak{q}} y$, then $\operatorname{ord}_{\mathfrak{q}}(x+y) = \operatorname{ord}_{\mathfrak{q}}(x)$

Diophantine Sets or Existentially Definable Sets

Let R be a commutative integral domain. A subset $A \subset R^m$ is called Diophantine over R if there exists a polynomial $p(T_1, \ldots T_m, X_1, \ldots, X_k)$ with coefficients in R such that for any element $(t_1, \ldots, t_m) \in R^m$ we have that

$$\exists x_1, \ldots, x_k \in R : p(t_1, \ldots, t_m, x_1, \ldots, x_k) = 0$$

$$\Updownarrow$$

$$(t_1, \ldots, t_m) \in A.$$

In this case we call $p(T_1, \ldots, T_m, X_1, \ldots, X_k)$ a Diophantine definition of A over R.

Integrality at Finitely Many Primes When the Field of Constants is Finite

Proposition (Robert Rumely, 1980)

If K is a function field over a finite field of constants, $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ is a finite collection of primes of K, then the set $\{x \in K : \mathrm{ord}_{\mathfrak{p}_i} x \geq 0, i = 1, \ldots, m\}$ is existentially definable over K.

Outline

p-divisibility

Definition

Let $x, y \in \mathbb{Z}_{\neq 0}$ and let p be a rational prime. In this case we will say that $x|_p y$ if $y = xp^s$, where $s \in \mathbb{Z}_{\geq 0}$.

Proposition (Pheidas 1987)

If p is a rational prime, then multiplication is existentially definable in the system $(\mathbb{Z}_{>0}, +, |_p)$.

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Simulating Integers with Multiplication

What does this mean?

The exist linear polynomials

$$L_i(T_1, T_2, T_3, X_1, \ldots, X_m),$$

$$M_i(T_1, T_2, T_3, X_1, \ldots, X_m),$$

$$N_i(T_1, T_2, T_3, X_1, \ldots, X_m),$$

with coefficients in $\mathbb{Z}$ and with $i = 1 \ldots, n$ such that for any positive integers a_1, a_2, a_3 the system

$$\begin{cases} L_i(a_1, a_2, a_3, X_1, \ldots, X_m) |_p M_i(a_1, a_2, a_3, X_1, \ldots, X_m), \\ N_i(a_1, a_2, a_3, X_1, \ldots, X_m) = 0, \\ i = 1, \ldots, n \end{cases}$$

has solutions in positive integers if and only if $a_3 = a_2 a_1$.

An Undecidability Consequence

Corollary

There is no algorithm to decide whether an arbitrary system of the form

$$\begin{cases} L_i(X_1, \ldots, X_r) |_p M_i(X_1, \ldots, X_r), \\ N_i(X_1, \ldots, X_r) = 0, \\ i = 1, \ldots, \ell \end{cases}$$

where

$$L_i(X_1, \ldots, X_r),$$

$$M_i X_1, \ldots, X_r),$$

$$N_i(X_1, \ldots, X_r),$$

are linear polynomials with coeffcients in $\mathbb{Z}$, has solutions in positive integers.

Conmnecting to Diophantine Undecidability over Function Fields

Proposition

Let K be a countable function field over a field of constants k of positive characteristic p. Let $\mathfrak{q}$ be a prime of K. Suppose the following subsets of K are Diophantine over K:

$$INT = \{x \in K : \mathrm{ord}_{\mathfrak{q}}\, x \geq 0\};$$

$$p(K) = \{(x, y) \in K^2 : y = x^{p^s}, s \in \mathbb{Z}_{\geq 0}\}.$$

Then HTP is unsolvable over K.

Constructing a Model of $(\mathbb{Z}_{\geq 0}, +, |_p)$

Proof.

Send $n \longrightarrow A_n = \{x \in K : \mathrm{ord}_{\mathfrak{q}}\, x = n\}$. Observe the following:

- For any $x \in K$ we have that $\exists n : x \in A_n \Leftrightarrow \mathrm{ord}_{\mathfrak{q}}\, x \geq 0$
- $x, y \in A_n \Leftrightarrow \mathrm{ord}_{\mathfrak{q}}\, \frac{x}{y} = 0$
- $x \in A_n, y \in A_m, z \in A_{n+m} \Leftrightarrow \mathrm{ord}_{\mathfrak{q}}\, \frac{xy}{z} = 0$
- $x \in A_n, y \in A_m, n|_p m \Leftrightarrow \exists s \in \mathbb{Z}_{\geq 0}, \exists z \in A_n : y = z^{p^s}$

Integrality at a Prime for the Transcendence One Degree Case

Theorem (S. 2000)

If K is a function field of positive characteristic and transcendence degree one not containing the algebraic closure of a finite field, then for any prime $\mathfrak{q}$ of K the set

$$INT = \{x \in K : \mathrm{ord}_{\mathfrak{q}}\, x \geq 0\}$$

is existentially definable over K.

Integrality at a Prime for the Higher Transcendence Degree Case

Theorem (Eisentraeger, S., work in progress)

If K is a function field of positive characteristic and not containing the algebraic closure of a finite field, then for some prime $\mathfrak{q}$ of K there exists a set $I \subset K$ such that

- *I is Diophantine over K.*
- *If $x \in I$, then $\operatorname{ord}_{\mathfrak{q}} x \geq 0$.*
- *If $x \in \mathbb{Z}/p(t)$, and $\operatorname{ord}_{\mathfrak{q}} x \geq 0$, then $x \in I$.*

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Outline

p-th Powers Are Definable Everywhere

Theorem (The New Result on p-th Powers)

Let K be any function field of positive characteristic p. In this case the set

$$p(K) = \{(x, x^{p^n}) : x \in K, n \in \mathbb{Z}_{\geq 0}\}$$

is existentially definable over K. (Joint work with Kirsten Eisentraeger)

The General Plan

Notation

- Let k be a field of characteristic $p > 0$,
- let t be transcendental over k,
- let K be a finite separable extension of $k(t)$.

The Three Step Program

1. Define p-th powers of t.
2. Define p-th powers of a set of functions with simple zeros and poles.
3. Define p-th powers of arbitrary functions.

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p-th Powers of t over Rational Function Field of Characteristic Greater Than 2

Lemma (Pheidas)

Let k be a finite field of characteristic $p > 2$. Let t be transcendental over k. Then the equations below are satisfied with $u, v, w \in k(t)$ if and only if for some $s \in \mathbb{Z}_{\geq 0}$ we have that $w = t^{p^s}$.

$$\begin{cases} w - t = v^p - v \\ \frac{1}{w} - \frac{1}{t} = u^p - u \end{cases} \tag{1}$$

Satisfiability is easy

For any $x \in K$ and any $s \in \mathbb{Z}_{\geq 0}$, since in characteristic $p > 0$ it is the case that $(a+b)^p = a^p + b^p$, we have

$$x^{p^s} - x = (x^{p^{(s-1)}} + x^{p^{(s-2)}} + \ldots + x)^p - (x^{p^{(s-1)}} + x^{p^{(s-2)}} + \ldots + x) \tag{2}$$

p-th Powers of t over Rational Function Field of Characteristic Greater Than 2

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Conctructing p-th powers of t

We proceed in two steps. First we show that if w satisfies equations below, then it is equal to t or it is a p-th power.

$$\begin{cases} w - t = v^p - v \\ \frac{1}{w} - \frac{1}{t} = u^p - u \end{cases} \tag{3}$$

Second, we show that if $w = w_1^p$ we can rewrite the equations above:

$$\begin{cases} w_1 - t = (v^p - w_1^p) + (w_1 - v) = v_1^p - v_1 \\ \frac{1}{w_1} - \frac{1}{t} = u^p - \frac{1}{w_1^p} + \frac{1}{w_1} - u = u_1^p - u_1 \end{cases} \tag{4}$$

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The Denominators of $v^p - v$ and $w - t$ in a rational field.

Suppose $v = \frac{A}{z_2}$, where A, z_2 are relatively prime polynomials. In this case $v^p - v = \frac{A^p}{z_2^p} - \frac{A}{z_2^p} = \frac{A^p - Az_2^{p-1}}{z_2^p}$. Observe that $(A^p - Az_2^{p-1}, z_2^p) = 1$ as polynomials over k. Indeed, if P is a prime polynomial dividing z_2^p, then P divides z_2 and P is prime to A, and therefore to $A^p - Az_2^{p-1}$. Thus z_2^p is the reduced denominator of $v^p - v$.

We now have $w - t = v^p - v = \frac{a}{z_2^p}$, where z_2, a are relatively prime polynomials. Since t does not have a denominator, we conclude that $w = \frac{Z_1}{z_2^p}$, where z_2, Z_1 are relatively prime polynomials.

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The numerator of w.

We now consider the second equation $\frac{1}{w} - \frac{1}{t} = u^p - u$ and by a similar argument conclude that $\frac{1}{w} - \frac{1}{t} = \frac{b}{z_1^p}$. Thus,

$$\frac{z_2^p}{Z_1} - \frac{1}{t} = \frac{tz_2^p - Z_1}{tZ_1} = \frac{b}{z_1^p}.$$

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If $k > 1$, then $(z_2^p, t^{k-1}\tilde{Z}_1) = 1$ and Z_1 is the reduced denominator forced to be equal to z_1^p. If $k = 1$, then the numerator is equal to $z_2^p - \tilde{Z}_1$ and may be divisible by t, so that $\frac{Z_1}{t} = z_1^p$. In other words, either $w = t\frac{z_1^p}{z_2^p}$ or $w = \frac{z_1^p}{z_2^p}$. In the last case w is a p-th power.

Why is w a p-th power in any case?

If w is not a p-th power, then $w = t\frac{z_1^p}{z_2^p}$, where z_1, z_2 are relatively prime polynomials and it satisfies an equation $w - t = v^p - v = \frac{A}{z_2^p} - \frac{A}{z_2}$. Thus, $z_2 v$ is also a polynomial in t.

Now we rewrite the equation in the following form:

$$tz_1^p - z_2^p t = (z_2 v)^p - (z_2 v) z_2^{p-1}$$

$$tz_1^p - (z_2 v)^p = z_2^p t - (z_2 v) z_2^{p-1}$$

Looking at the right side observe that that any prime polynomial dividing z_2 occurs to the power at least 2 on the right. So if we differentiate the left side with respect to t we should get a polynomial which has all the zeros of z_2. However the derivative of the left side is z_1^p leading us to the conclusion that z_2 is a constant.

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Degree of Prime

Definition (Degree of a Prime)

If $\mathfrak{p}$ is a prime of a function field K over a field of constants k, we consider an equivalence relation on elements of K defined as follows: $x \equiv y \mod \mathfrak{p} \iff \mathrm{ord}_{\mathfrak{p}}(x-y) > 0$. The equivalence classes with respect to this relation form a field, called residue field of $\mathfrak{p}$. This field is isomorphic (structurally equivalent) to a finite extension of the constant field k. The degree of this extension is called the degree of the prime.

Definition

If K is a function field, then a divisor is a formal product $\prod_{\mathfrak{p}} \mathfrak{p}^{a(\mathfrak{p})}$, where the product is taken over all the primes of K, $a(\mathfrak{p}) \in \mathbb{Z}$ and all but finitely many exponents are zero. If $x \in K$, then the divisor of x is

$$(x) = \prod \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}} x}$$

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Principal Divisors and Class Number

Definition

A divisor is principal if it is a divisor of a field element. A class number (if it is exists) can be defined as the size of the group of divisors with the same degree of "zeros" and "poles" modulo the group of principal divisors. Any divisor with the same degree of "zeros" and "poles" raised to the class number becomes principal.

Remarks on the General Case

The preceding argument works for a rational function field over any field of constants of characteristic greater than 2, but it relies on the following facts:

- the class number of a rational field is one;
- t has a divisor of the form $\frac{p}{q}$ (i.e one pole and one zero, both of degree 1);
- the positive order at a prime goes down by at most one under differentiation.

To overcome these difficulties we usually have to establish that the divisor not just of w is a p-th power of another divisor but also that the divisors of $w + a$ for sufficiently many constants a are p-th powers of other divisors. We also need to use a consequence of Riemann-Roch theorem to show that if w does not have a divisor which is a p-th power of another divisor, then it can be of bounded height only with the bound depending on the genus of the field. If the height of w is bounded, then we can "push" it into the rational subfield where things proceed essentially as above.

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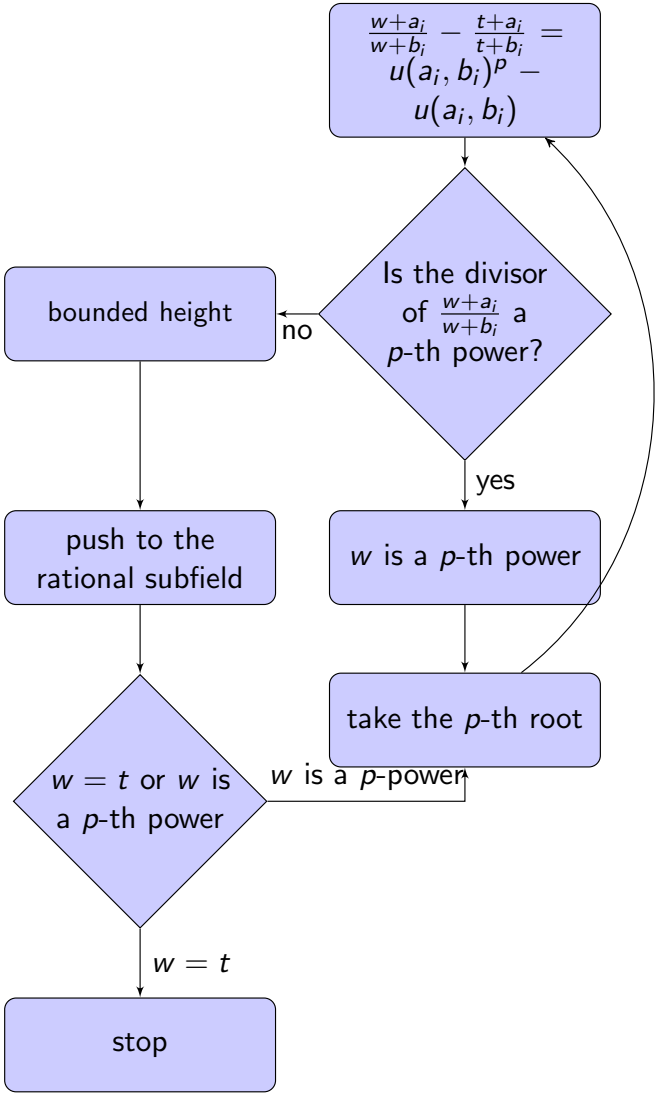
Flowchart


$\frac{w+a_i}{w+b_i} - \frac{t+a_i}{t+b_i} = u(a_i, b_i)^p - u(a_i, b_i)$
Is the divisor of $\frac{w+a_i}{w+b_i}$ a p-th power?
no
bounded height
yes
push to the rational subfield
w is a p-th power
take the p-th root
$w = t$ or w is a p-th power
w is a p-power
$w = t$
stop