A Computational Interpretation of the Axiom of Determinacy in Arithmetic

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Introduction

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- ►Can we understand everything from a computational viewpoint?

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"computational model" of the **axiom of determinacy** (AD) in classical arithmetic

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- The axiom of determinacy
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- A realizer of AD
- More on this model
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The axiom of determinacy (1/4)

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Definition 1. Player I **wins** in $G(A)$ if

$$\langle x_0, y_0, x_1, y_1, \ldots \rangle \in A.$$

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A **strategy** for Player I is a function

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The **axiom of determinacy** (AD):

Every $A \subset \omega^\omega$ is determined.

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- Determinacy in arithmetic:
  Lots of work on its strength both in 2^ω and in ω^ω in the area of classical reverse mathematics

Systems of arithmetic & Negative translation

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- $HA^{\omega}_{-} \cdots$ base logic is minimal
- $HA^{\omega}_{c} \cdots$ base logic is classical

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Proposition 1. (BBC 98) $HA^\omega_c + DC \vdash \phi$ implies $HA^\omega_- + DC^K \vdash \phi^K$, where K is a negative translation ($\neg\neg$ in front of all prime and $\exists$).

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$\mathscr{P}$ enjoys **syntactic continuity**:

For any $\chi : (\mathtt{N} \to \mathtt{N}) \to \mathtt{N}$, $f : \mathtt{N} \to \mathtt{N}$ and $g : \mathtt{N} \to \mathtt{N}$, if f and g are "sufficiently close", then $\chi f = \chi g$.

Realizability interpretation (2/3)

Definition 4. (The relation "$t : |\phi| \circledR \phi$")

- $t : \texttt{Abs} \circledR \perp$ if $t = \texttt{Axiom}_\texttt{i}\texttt{k}$
- $t : \texttt{Unit} \circledR t_1 = t_2$ if $t = ()$ and both t_1 and t_2 reduce to the same numeral in $\mathscr{P}$
- $t : |\phi_1| \to |\phi_2| \circledR \phi_1 \Rightarrow \phi_2$ if $tu \circledR \phi_2$ for all $u \circledR \phi_1$
- $t : |\phi_1| \times |\phi_2| \circledR \phi_1 \wedge \phi_2$ if $t = \langle t_1, t_2 \rangle$ and $t_i \circledR \phi_i$
- $t : \tau \to |\phi| \circledR \forall x : \tau\ \phi$ if $tu \circledR \phi[u/x]$ for all reducible $u : \tau$
- $t : \tau \times |\phi| \circledR \exists x : \tau\ \phi$ if $t = \langle p, u \rangle$ with $p : \tau$ reducible and $u \circledR \phi[p/x]$

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Remark 1. From Proposition 1 and the above theorem, we find that ϕ^{K} is realizable for every theorem ϕ of $HA^{\omega}_{c} + DC$.

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- $\mathbf{OP}(\chi) := \forall f \ \{\chi(f) = 1 \Rightarrow \exists m \ \forall g \ (\mathtt{eq}_{\leq m}(f, g) = 1 \Rightarrow \chi(g) = 1)\}$

Remark 2. $OP(\chi)$ expresses "χ represents an open subset of ω^ω."

A realizer of AD (2/4)

- **I has a w.s. in G(χ)** $:=$
  $$\exists\sigma : \mathtt{N} \to \mathtt{N}\ \forall y : \mathtt{N} \to \mathtt{N}\ (\chi(\sigma * y) = \mathtt{1})$$
  where $\sigma * y : \mathtt{N} \to \mathtt{N}$ is
  $$\sigma{*}y\ (\mathtt{i}) := \begin{cases} \sigma(\langle\langle\sigma * y(\mathtt{0}), \ldots, \sigma * y(\mathtt{i} - \mathtt{1})\rangle\rangle) & (i : \text{even}) \\ y\ ((\mathtt{i} - \mathtt{1})/\mathtt{2}) & (i : \text{odd}) \end{cases}$$
- **Det(χ)** $:= \neg(I\ has\ a\ w.s.\ in\ G(\chi)) \Rightarrow$
  $(II\ has\ a\ w.s.\ in\ G(\chi))$
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Lemma 1. (Gale-Stewart)
$HA^{\omega}_{c} + DC \vdash \forall \chi\ \{OP(\chi) \Rightarrow Det(\chi)\}$.

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The absence of **comprehension schema**:

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The absence of **comprehension schema**:

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In this world, **every** function is continuous (represents open sets).

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Theorem 2. AD^K is realizable.

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Proof. By Lemma 1 and Theorem 1, $\forall\chi\ \{OP(\chi) \Rightarrow Det(\chi)\}^K$ is realizable. Lemma 2 says $\forall\chi\ OP(\chi)^K$ is realizable. $\square$

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Remark 3. It would be difficult to realize (negative translation of) **real** determinacy.

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Remark 4. $HA^{\omega}_{c} + DC \nvdash AC(\mathtt{N} \to \mathtt{N}, \tau)$ follows from the above Corollary, but it is well-known that $HA^{\omega}_{c} + DC \vdash AC(\mathtt{N}, \tau)$.

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cf. $(E - PA^\omega) + \bigcup_\tau Comp(\tau) + (b\text{-}AC) + DC \nvdash (AC! - qf)^{\mathtt{N}\to\mathtt{N},\mathtt{N}}$ (U. Kohlenbach, 1992)

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How our program for AD work?

Our realizer heavily reflect the proof of GS's thm. (Since everything is continuous in this world, AD essentially reduce to the proof of GS's thm.)

Behavior of the realizer (2/2)

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With the help of game-theoretical intuition, can we find a realizer which behaves **symmetrically**?

cf. The notion of backtracking in Coquand's game semantics.