Compositional Abstraction Techniques for Probabilistic Automata (PA)

Falak Sher & Joost Pieter Katoen

RWTH Aachen

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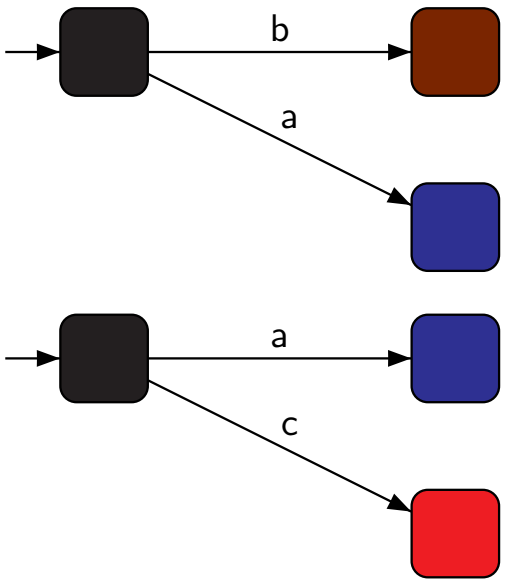
Abstraction of Labelled TS (LTS) with Modal TS (MTS)

LTS



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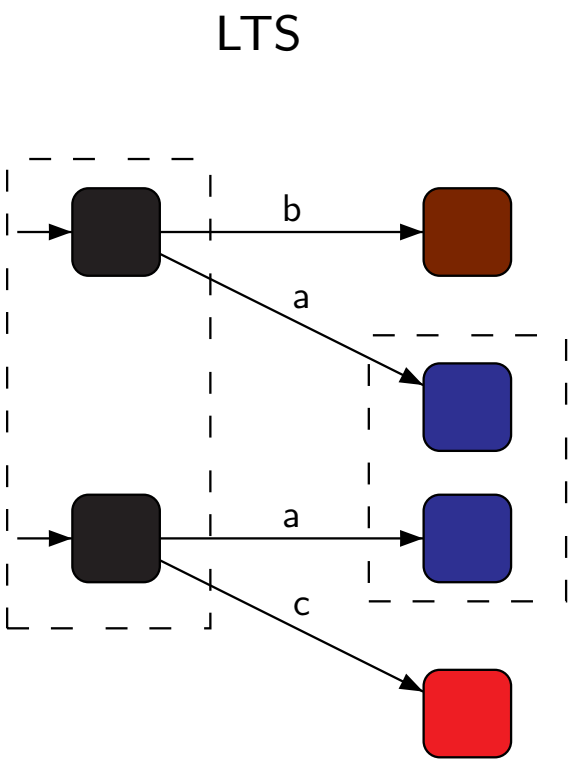

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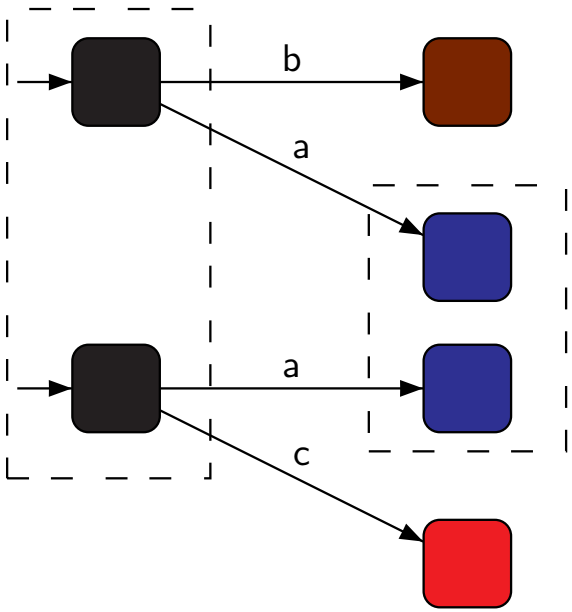
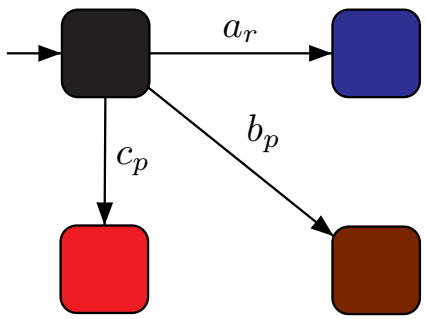


MTS

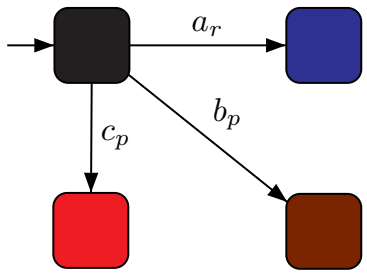


Modal Transition Systems: A Foundation for Three-Valued Program Analysis by Michael Huth, Radha Jagadeesan and David Schmidt. In: LNCS, Vol 2028. Springer (2001).

Contents

Analysis of a modal TS (MTS)

- Is it possible to reach a red state in N?



MTS

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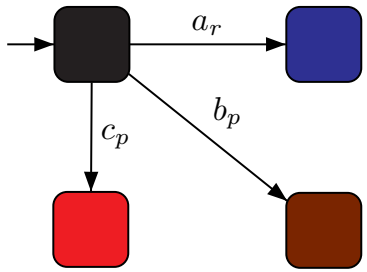

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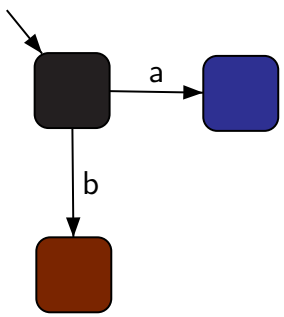
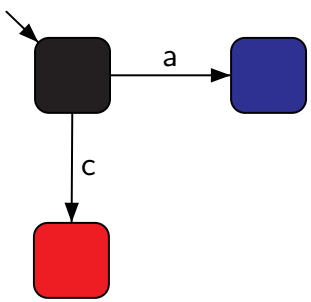
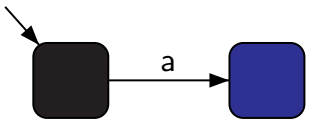
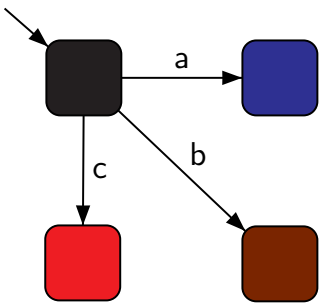
- Is it possible to reach a red state in each implementation of N?









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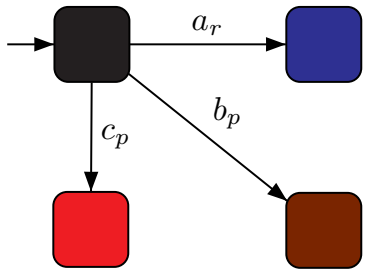

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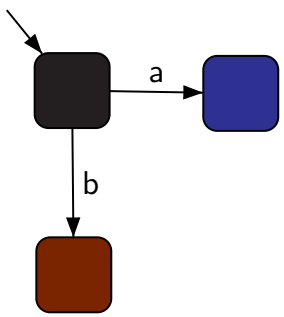

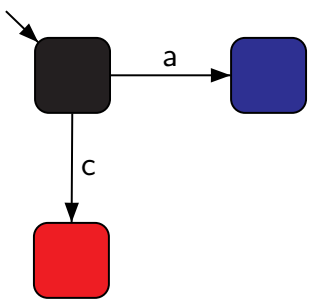

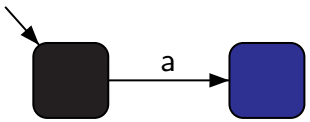

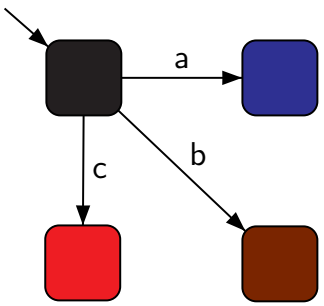

- **Don't know!!!**

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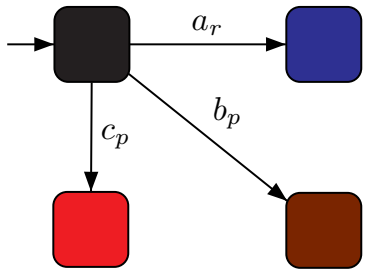

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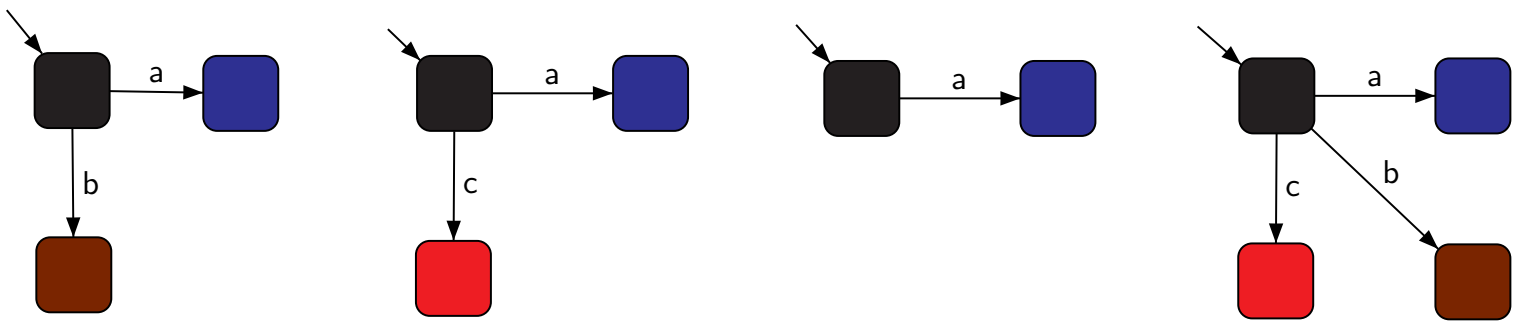

- **Don't know!!!**
- Is it possible to reach a blue state in N?.**Yes we can!!!**

Contents

Probabilistic automata (PA)

Definition

A *PA* is a tuple $N = (S, A, \Delta, s_0)$ where:

- S is a set of states, with $s_0 \in S$
- A is a set of actions, and
- $\Delta \subseteq S \times A \times Dist(S)$ is a set of transitions.

where $Dist(S)$ is a set of distributions over S.

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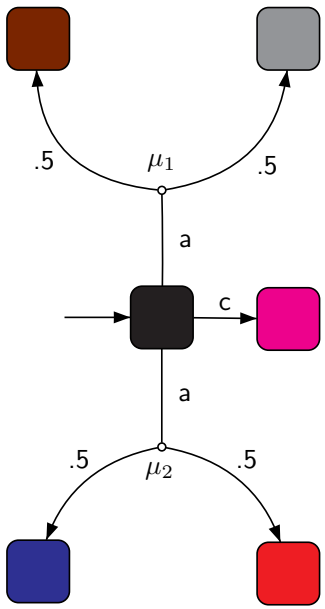

PA M_1

Modeling and Verification of Randomized Distributed Real-Time Systems by Roberto Segala (1995).

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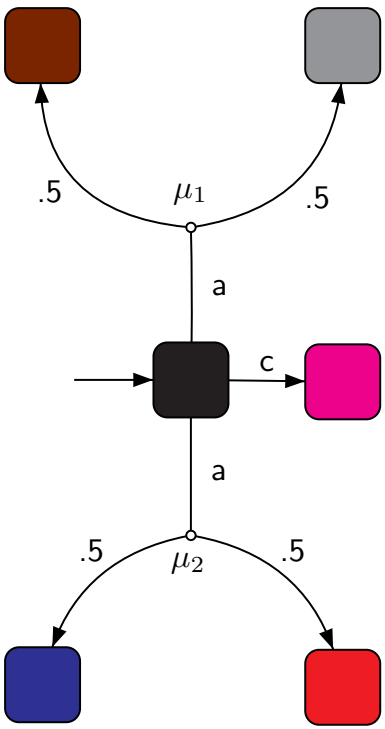
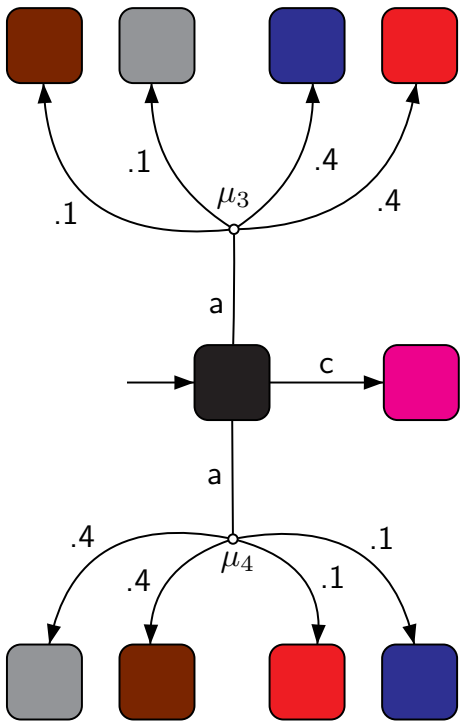
Abstraction of PA with modal PA (MPA)

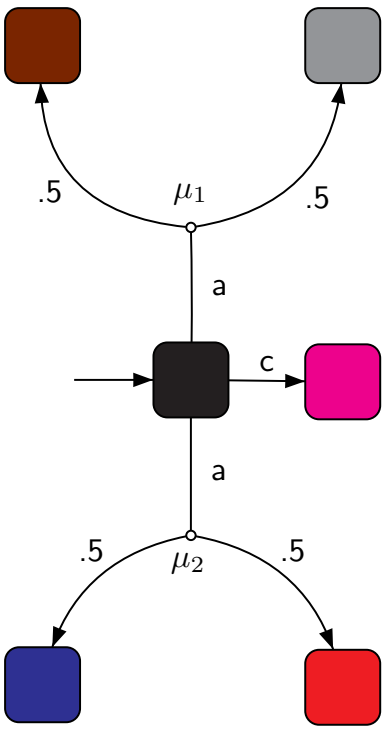


PA M_1

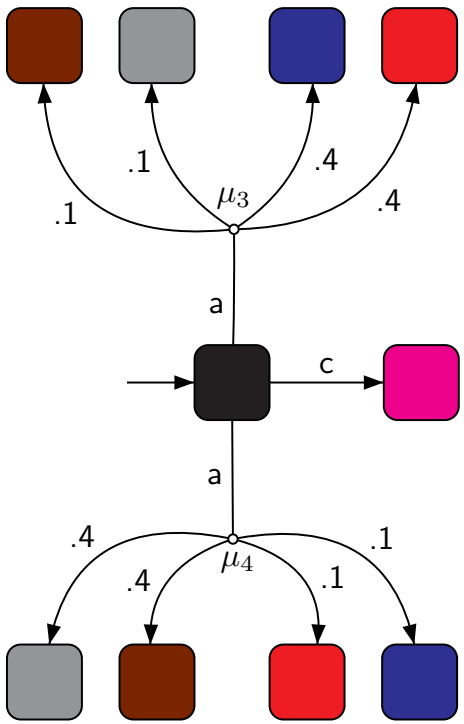


PA M_2

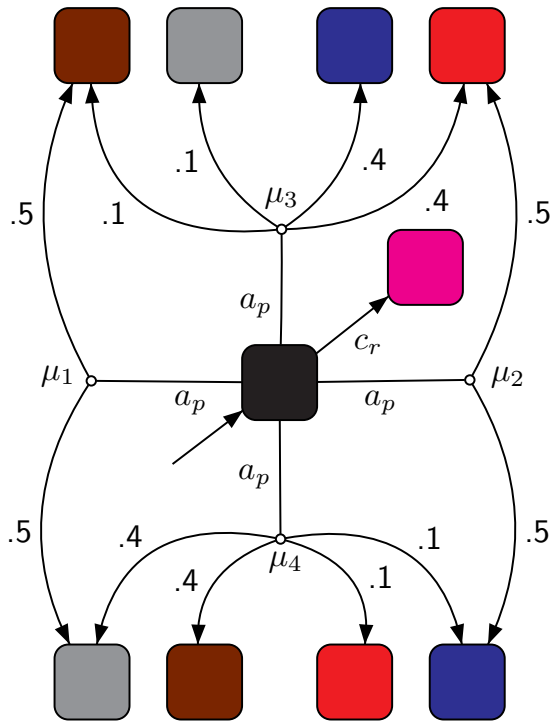
Abstraction of PA with modal PA (MPA)



PA M_1



PA M_2



Modal PA N

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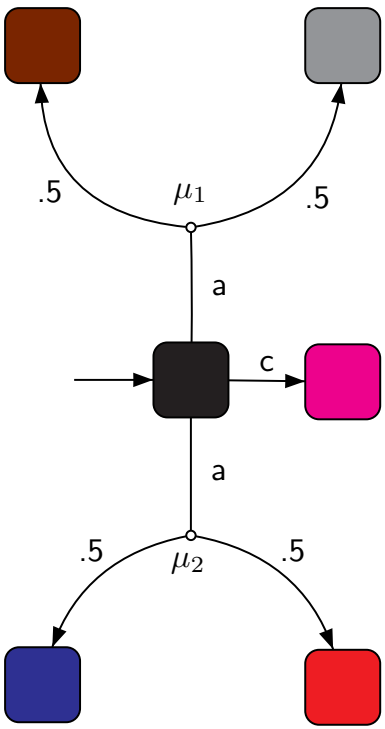

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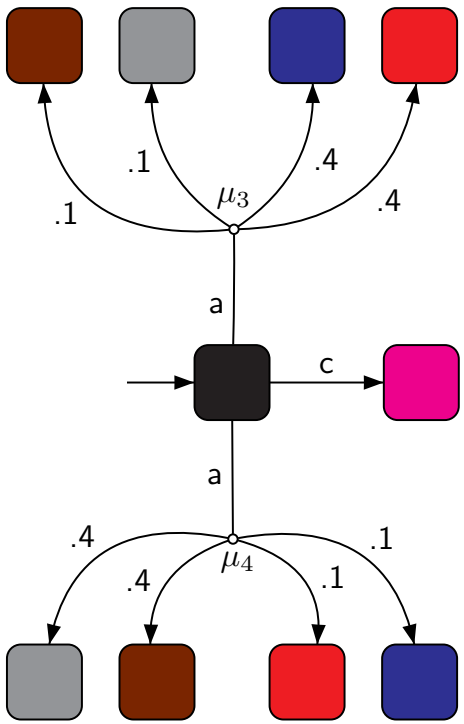
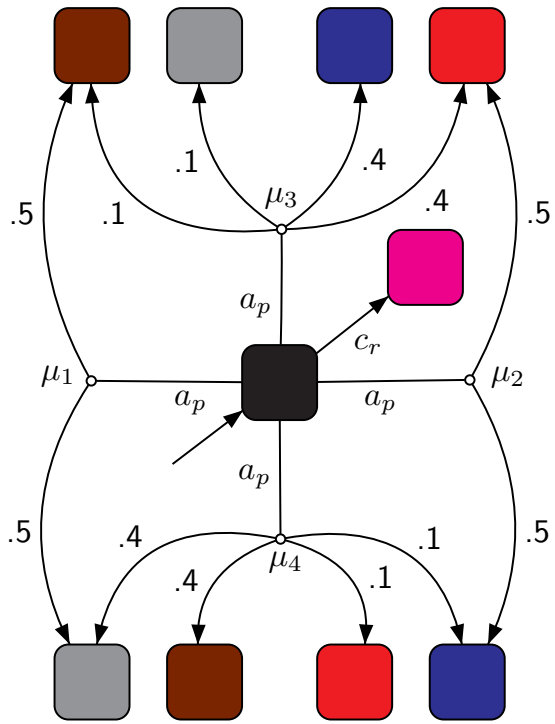


PA M_2



Modal PA N

Is it possible to reach a blue state with probability $\geq .1$? **Don't know!!!**.

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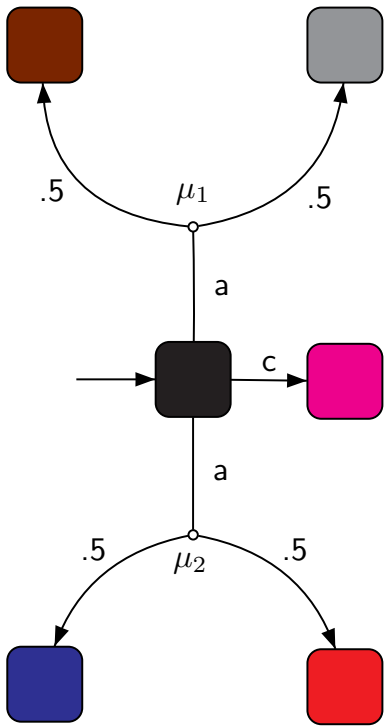
Simplification of PA



PA M_1

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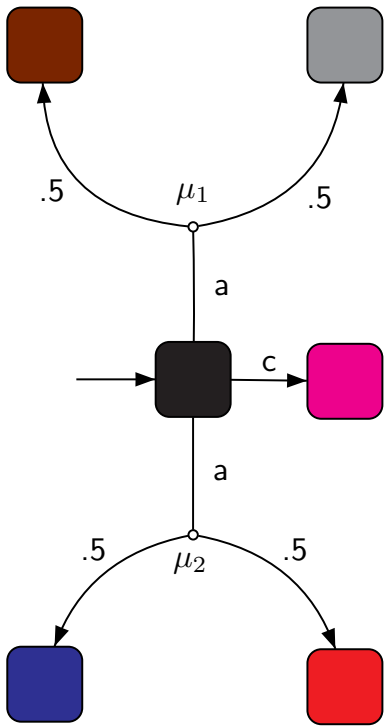
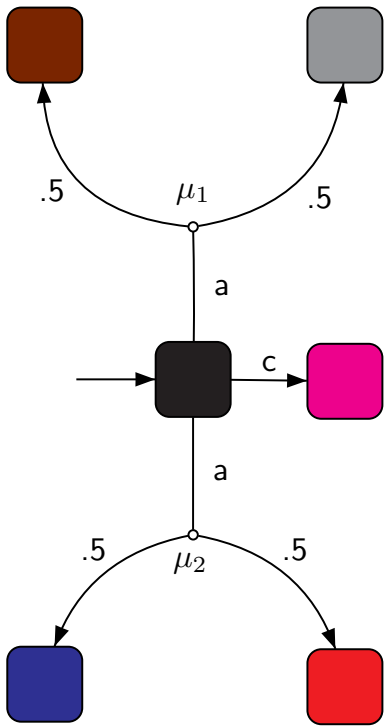


PA M_1

- Represent all combined a-transitions by one a-transition.

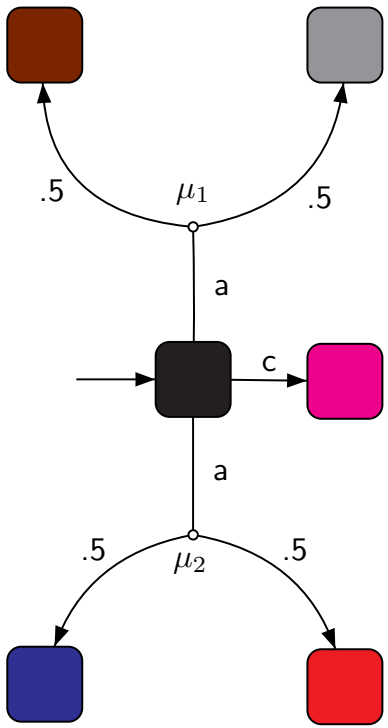
Simplification of PA



PA M_1

- Represent all combined a-transitions by one a-transition.
- let $\varphi_1 = \{\mu_1, \mu_2\}$ and $\varphi_1^C = \{c_1 \cdot \mu_1 \oplus c_2 \cdot \mu_2 \mid c_1 + c_2 = 1\}$

Simplification of PA



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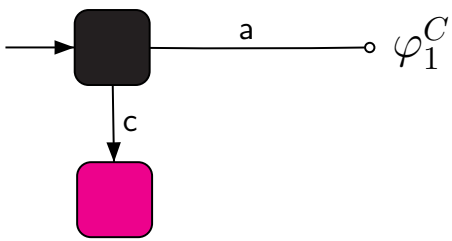
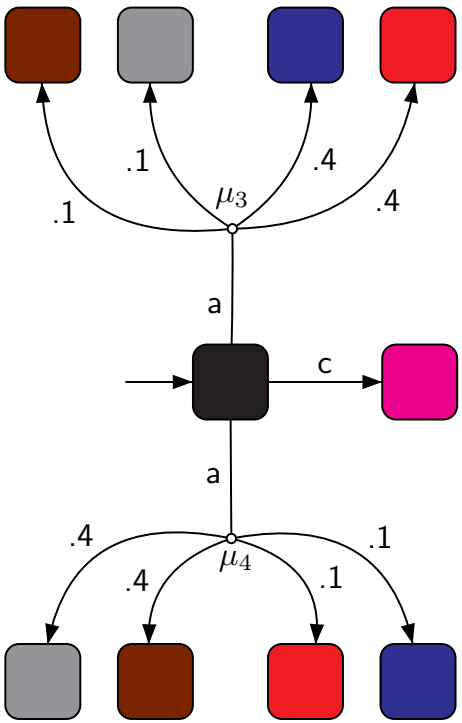


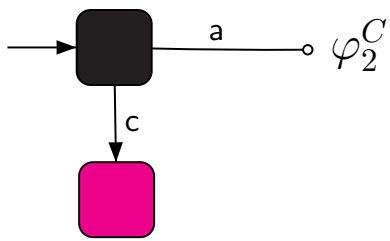
Constrained PA (CPA) N_1

Simplification of PA



PA M_2

- Represent all combined a-transitions by one a-transition.
- let $\varphi_2 = \{\mu_3, \mu_4\}$ and $\varphi_2^C = \{c_3 \cdot \mu_3 \oplus c_4 \cdot \mu_4 \mid c_3 + c_4 = 1\}$



Constrained PA (CPA) N_2

Abstraction of PA with abstract PA (APA)

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- $\mu_1 = [\![.5, .5]\!]$
- $\mu_2 = [\![.5, .5]\!]$
- $\varphi_1 = \{\mu_1, \mu_2\}$
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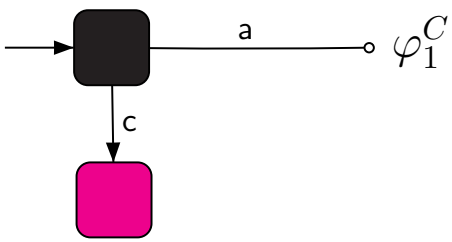

CPA N_1

Abstraction of PA with abstract PA (APA)

- $\mu_1 = [\![.5\blacksquare, .5\blacksquare]\!]$
- $\mu_2 = [\![.5\blacksquare, .5\blacksquare]\!]$
- $\varphi_1 = \{\mu_1, \mu_2\}$
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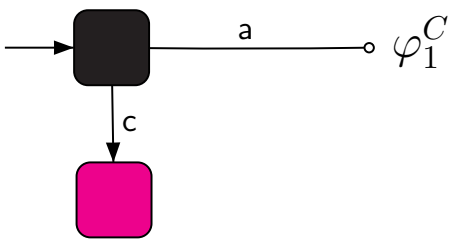
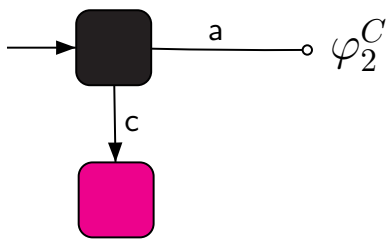


CPA N_1

- $\mu_3 = [\![.1\blacksquare, .1\blacksquare, .4\blacksquare, .4\blacksquare]\!] = .2\mu_1 \oplus .8\mu_2$
- $\mu_4 = [\![.4\blacksquare, .4\blacksquare, .1\blacksquare, .1\blacksquare]\!] = .8\mu_1 \oplus .2\mu_2$
- $\varphi_2 = \{\mu_3, \mu_4\}$
- $\varphi_2^C = \{c_3 \cdot \mu_3 \oplus c_4 \cdot \mu_4 \mid c_3 + c_4 = 1\}$



CPA N_2

Abstraction of PA with abstract PA (APA)

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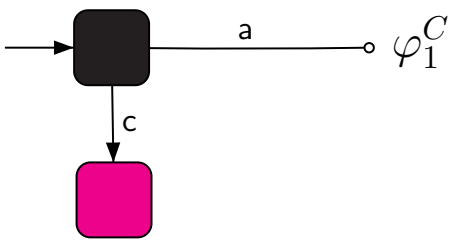
- $\mu_3 = [\![.1, .1, .4, .4]\!] = .2\mu_1 \oplus .8\mu_2$
- $\mu_4 = [\![.4, .4, .1, .1]\!] = .8\mu_1 \oplus .2\mu_2$
- $\varphi_2 = \{\mu_3, \mu_4\}$
- $\varphi_2^C = \{c_3 \cdot \mu_3 \oplus c_4 \cdot \mu_4 \mid c_3 + c_4 = 1\}$



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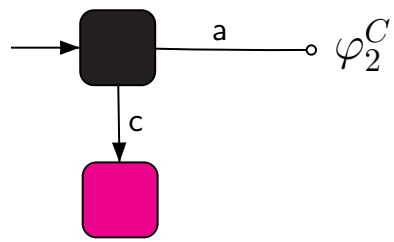
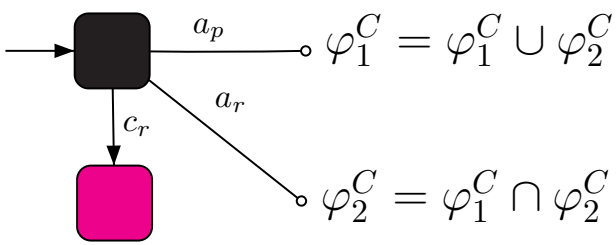


CPA N_2



APA N

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Abstract Probabilistic Automata (APA)

Definition

An *APA* is a tuple $N = (S, A, \Delta_r, \Delta_p, s_0)$ where S, A and s_0 are as before, and:

- $\Delta_r \subseteq S \times A \times C(S)$ is a set of *required* transitions,
- $\Delta_p \subseteq S \times A \times C(S)$ is a set of *possible* transitions with $\Delta_r \subseteq \Delta_p$,

$C(S)$ is a set of polynomial constraints.

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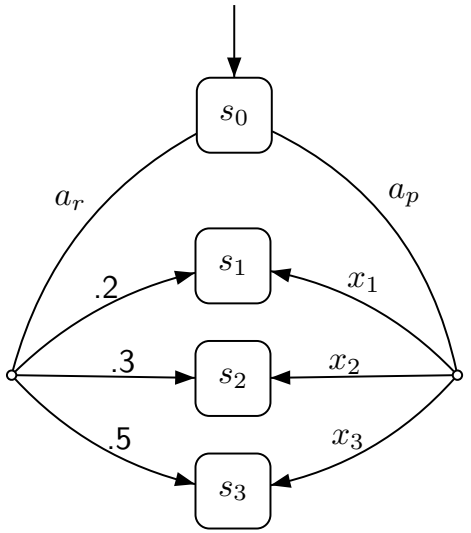


$\varphi_x = (x_1 = .4 \wedge x_2 = .6) \vee x_3 = 1$

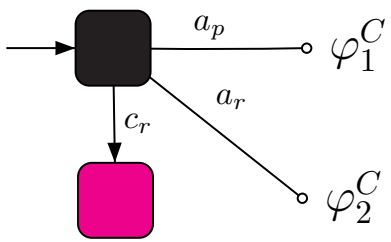
An APA

Abstract Probabilistic Automata by Benoît Delahaye, Joost-Pieter Katoen, Kim G. Larsen, Axel Legay, Mikkel L. Pedersen, Falak Sher and Andrzej Wasowski. In VMCAI . LNCS, Vol 6538. Springer (2011).

Contents

Probabilistic reachability in APA

- Is it possible to reach a blue state with probability $\geq .1$ in N?



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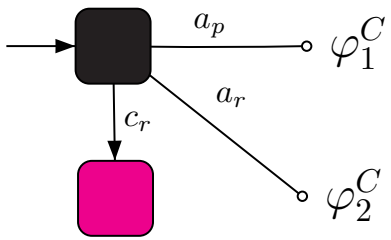

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Challenges in the analysis of APA

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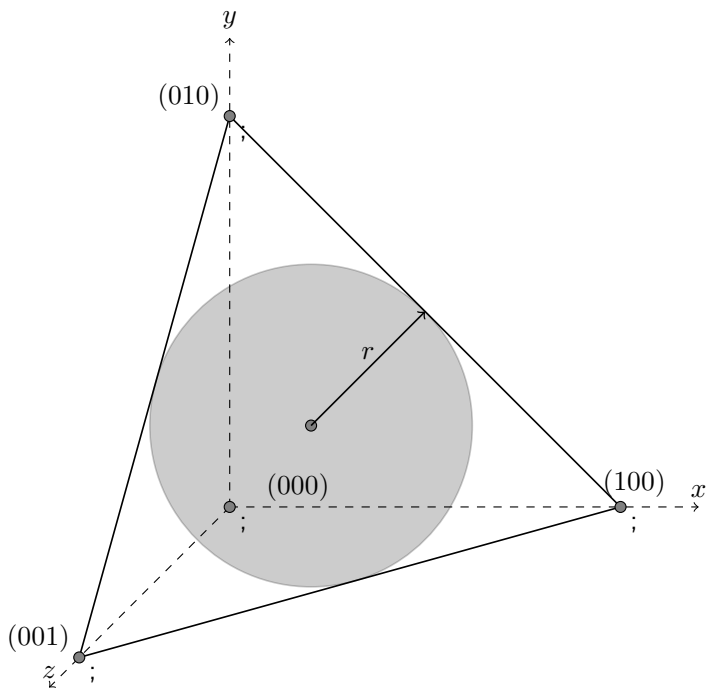
$$\varphi = (x^2 + y^2 + z^2 \leq r^2 \wedge x + y + z = 1)$$



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- **Challenge:** Infinite implementations are possible because of polynomial constraint functions.

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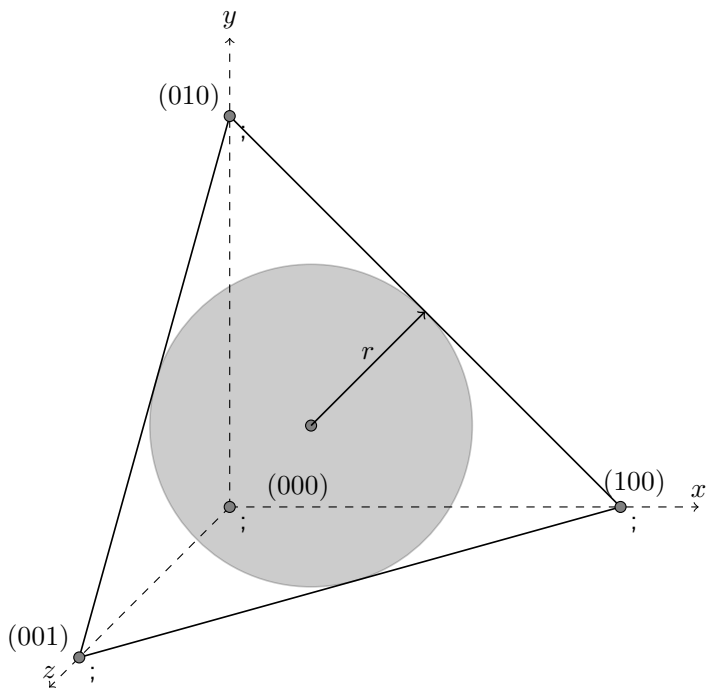

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  - **Step 1:** Approximate polynomial constraints by linear ones.

Challenges in the analysis of APA

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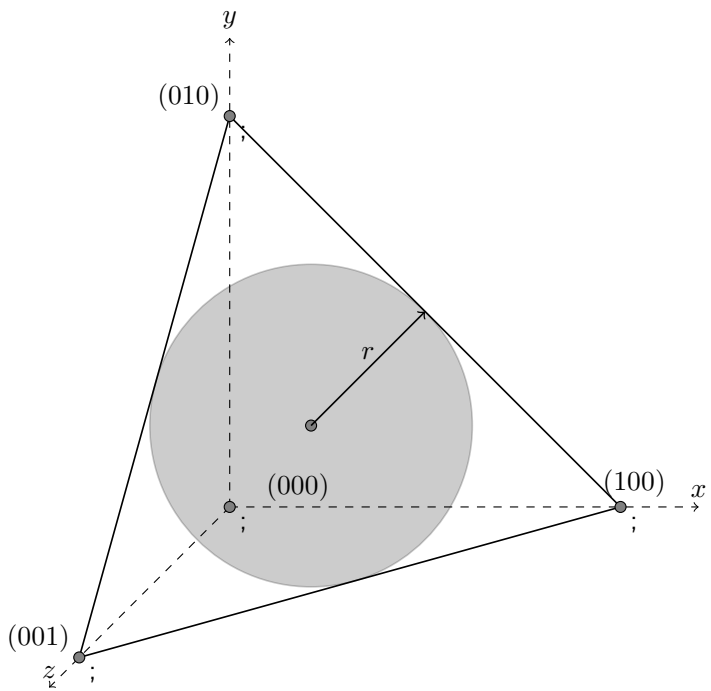

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Solution: (1) Approximation of a polynomial constraint function by a linear one

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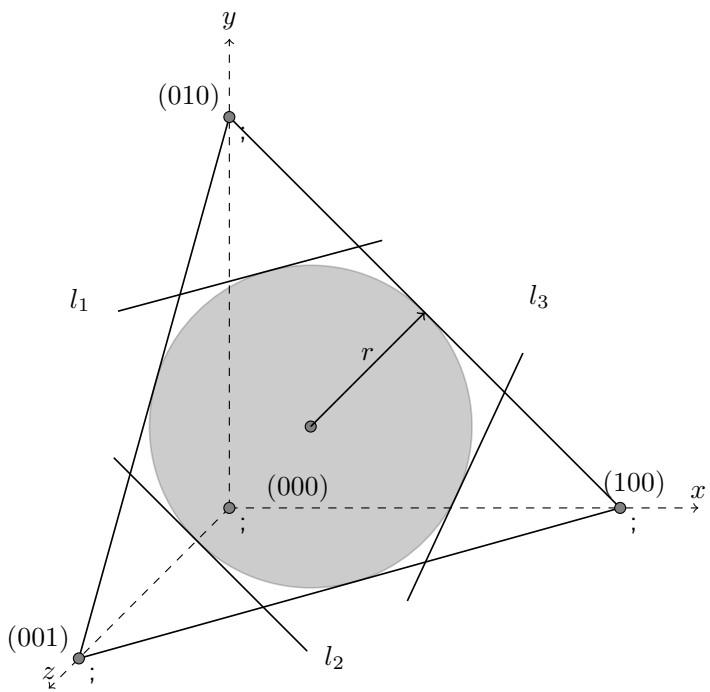
Over-approximation



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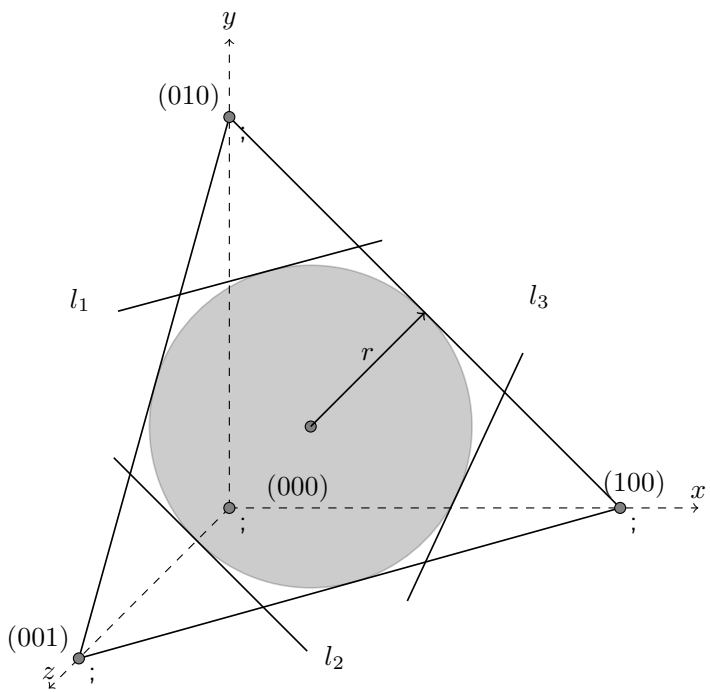

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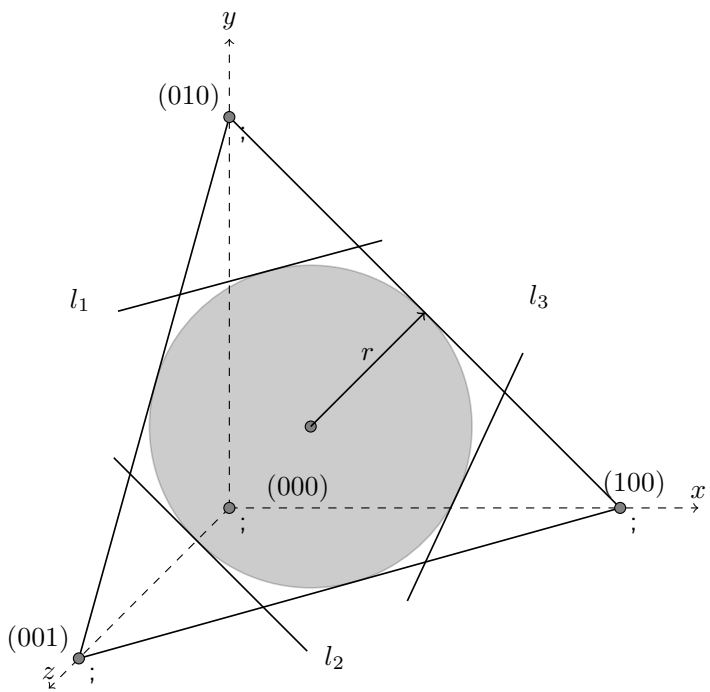

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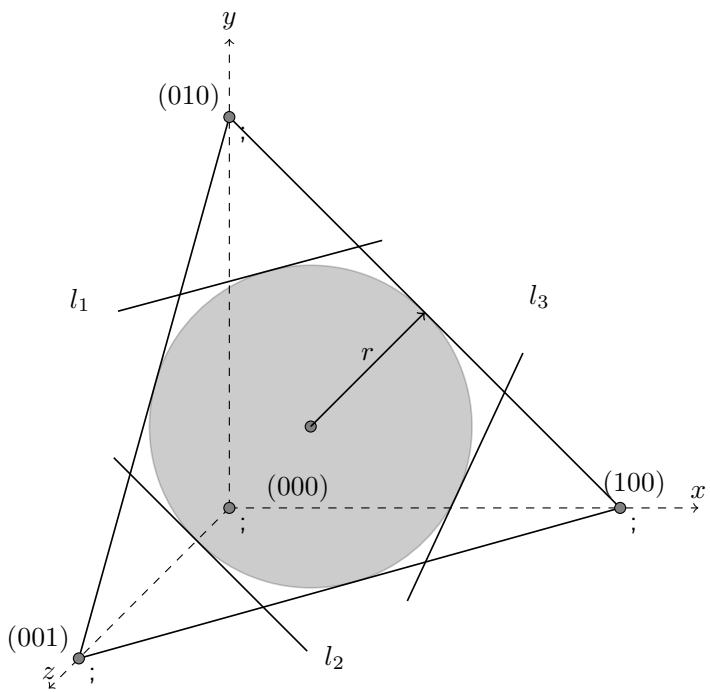

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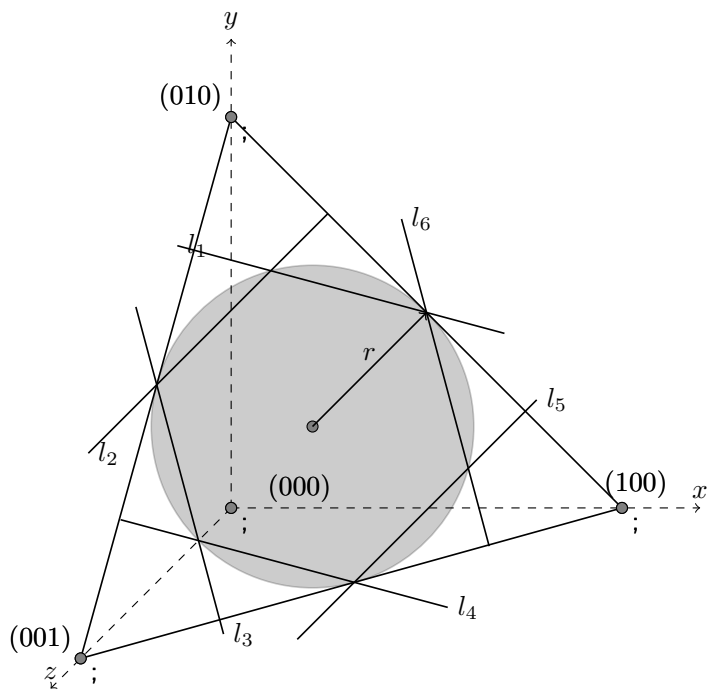

- $\varphi' = \{\mu \mid \mu \text{ is enclosed by } l_1 \text{ to } l_6\}$
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Solution: (1) Effect of constraint-approximation on APA

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Let N be an APA, then for every state s:

- if $s \xrightarrow{a}_r \varphi$, then under-approximate φ.

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Solution: (1) Effect of constraint-approximation on APA

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then for the resultant APA N':

- if PA M is an implementation of N, its also an implementation of N'.

Solution: (2) Exploitation of extreme distributions of linear constraints

- **Two step solution:**
  - **Step 1:** Approximate polynomial constraints by linear ones. **Done**
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- Let $\varphi = \{\eta_1, \eta_2, .2 \cdot \eta_1 \oplus .8 \cdot \eta_2\}$, then $\varphi_{extr} = \{\eta_1, \eta_2\}$.

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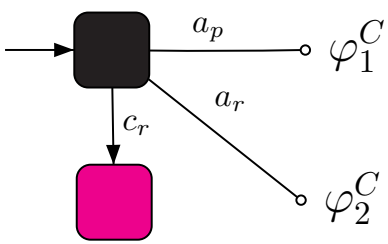
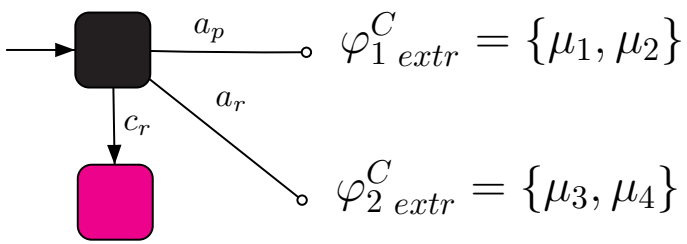
- For our running example:

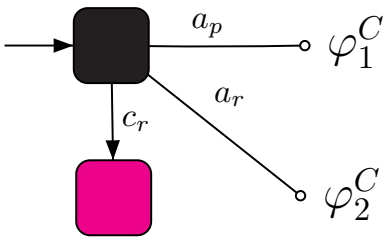




$$\varphi_1^C = \{c_1 \cdot \mu_1 \oplus c_2 \cdot \mu_2 \mid c_1 + c_2 = 1\}$$
$$\varphi_2^C = \{c_3 \cdot \mu_3 \oplus c_4 \cdot \mu_4 \mid c_3 + c_4 = 1\}$$

Probabilistic reachability in APA

- Is it possible to reach a blue state with probability $\geq .1$ in N?



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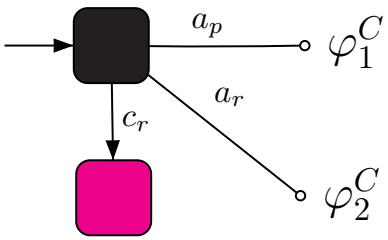

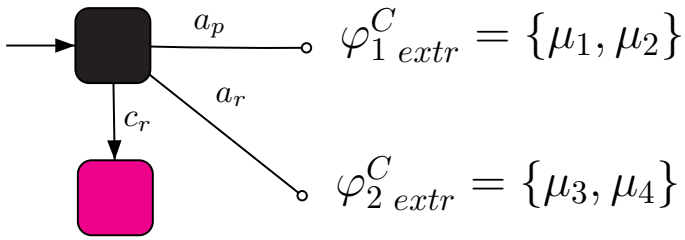

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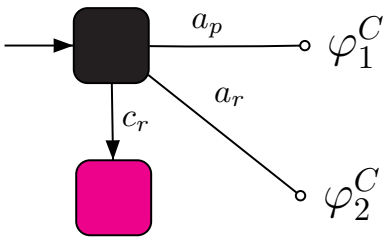
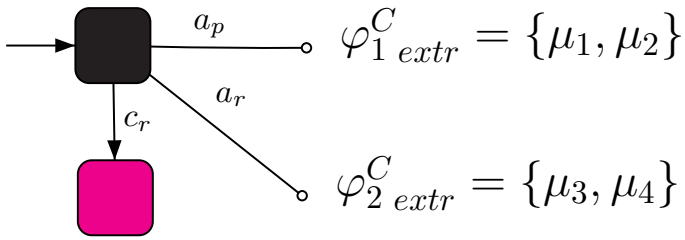
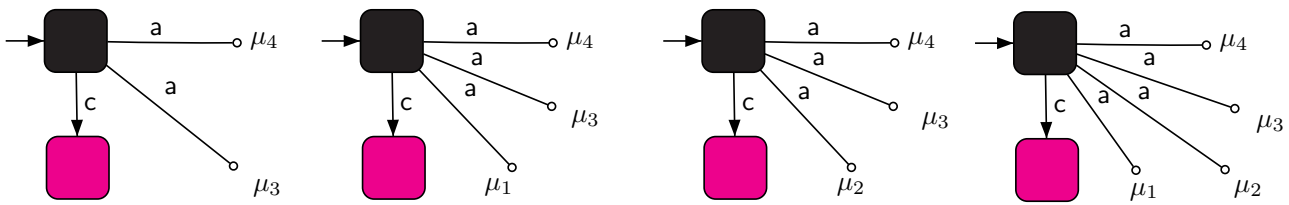




- Get all possible implementations of simplified APA:



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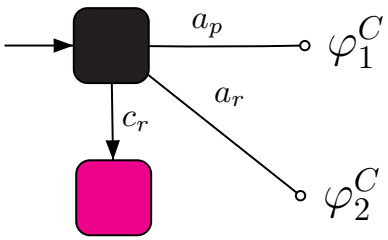

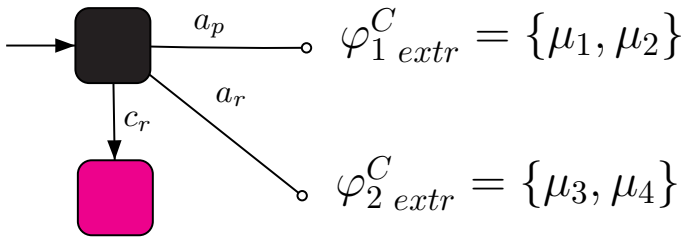

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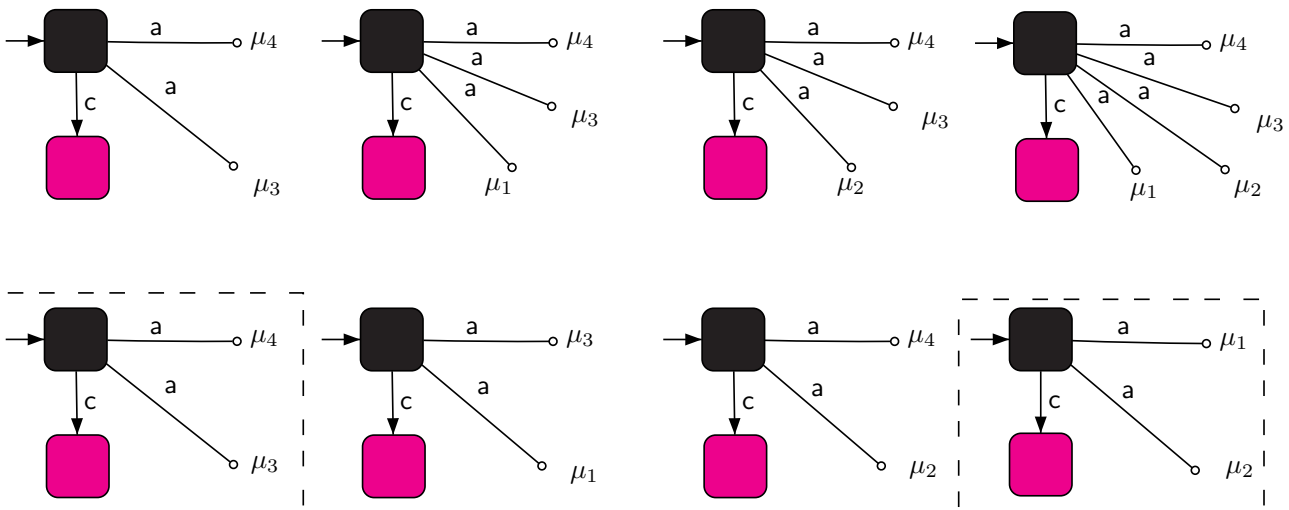

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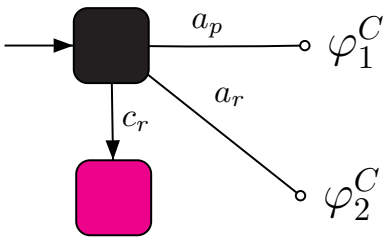

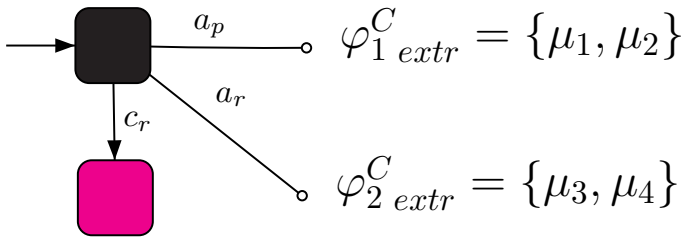

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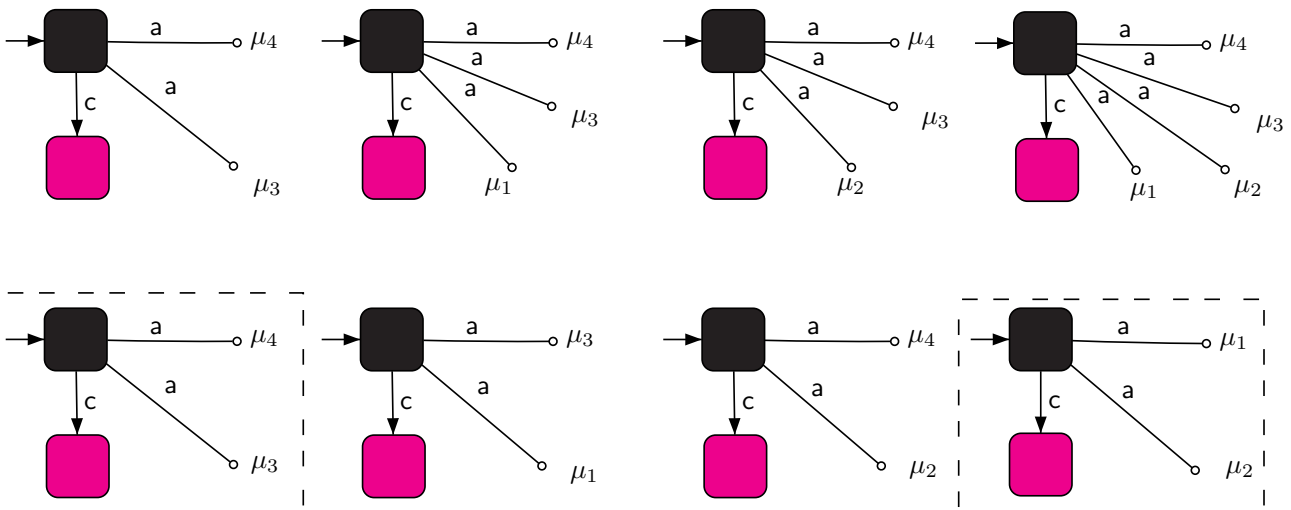

- $\mu_4(\blacksquare) = 0.1$ and $\blacksquare \xrightarrow{a} \mu_4$ is present in each implementation.

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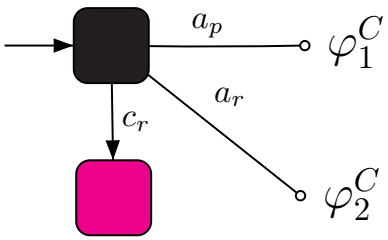

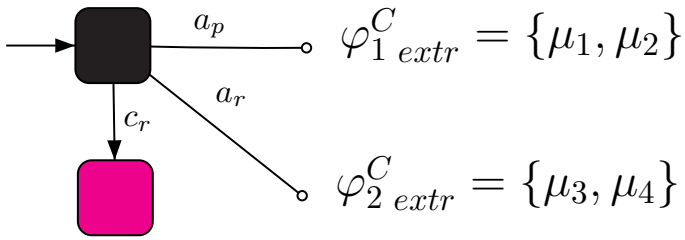

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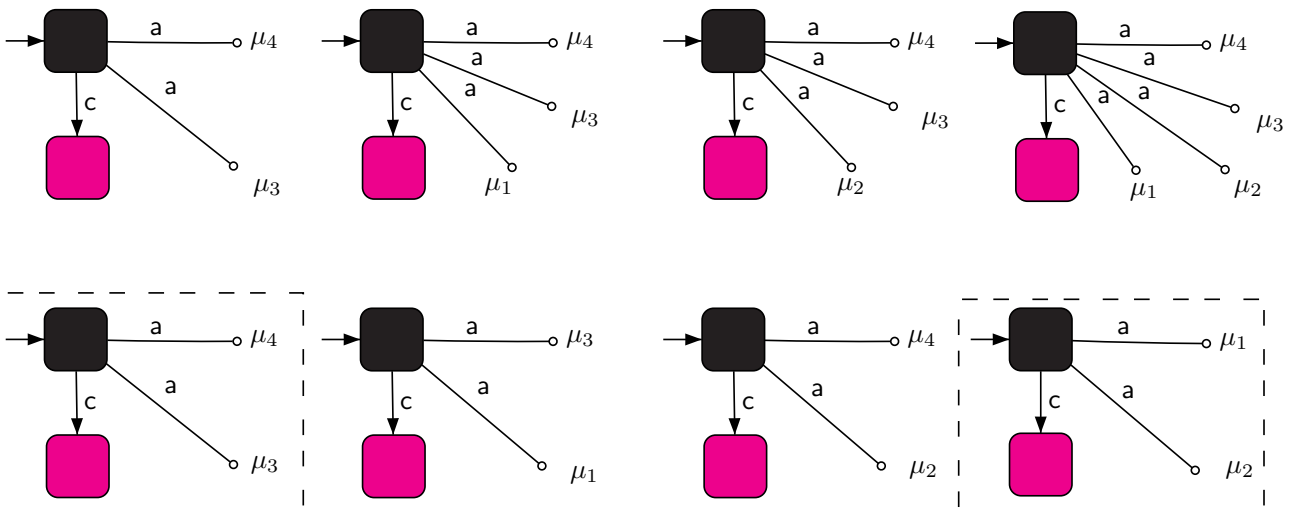

- $\mu_4(\blacksquare) = 0.1$ and $\blacksquare \xrightarrow{a} \mu_4$ is present in each implementation.
- **Yes, its possible!!!**

Lower/upper bounds of maximum reachability probabilities and extreme implementations of APA

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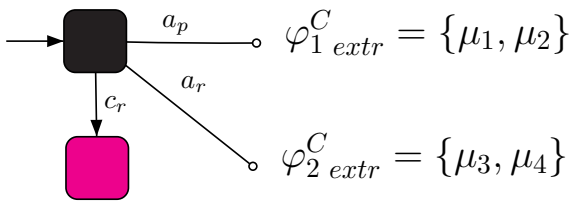
- Is the maximum probability to reach a blue state between $.4$ and $.5$ in N?



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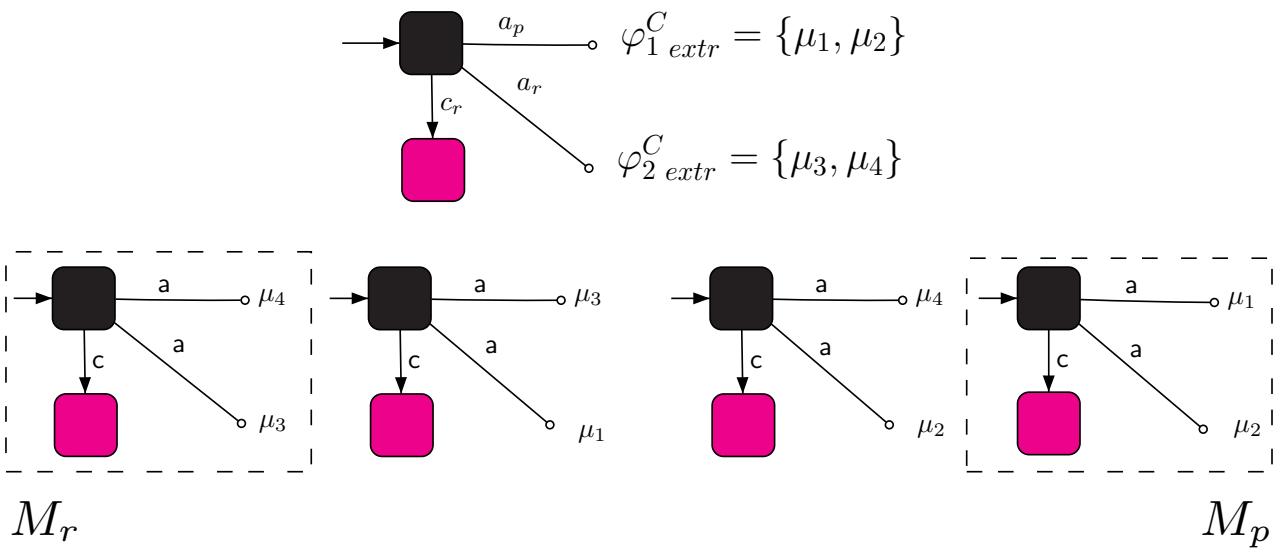

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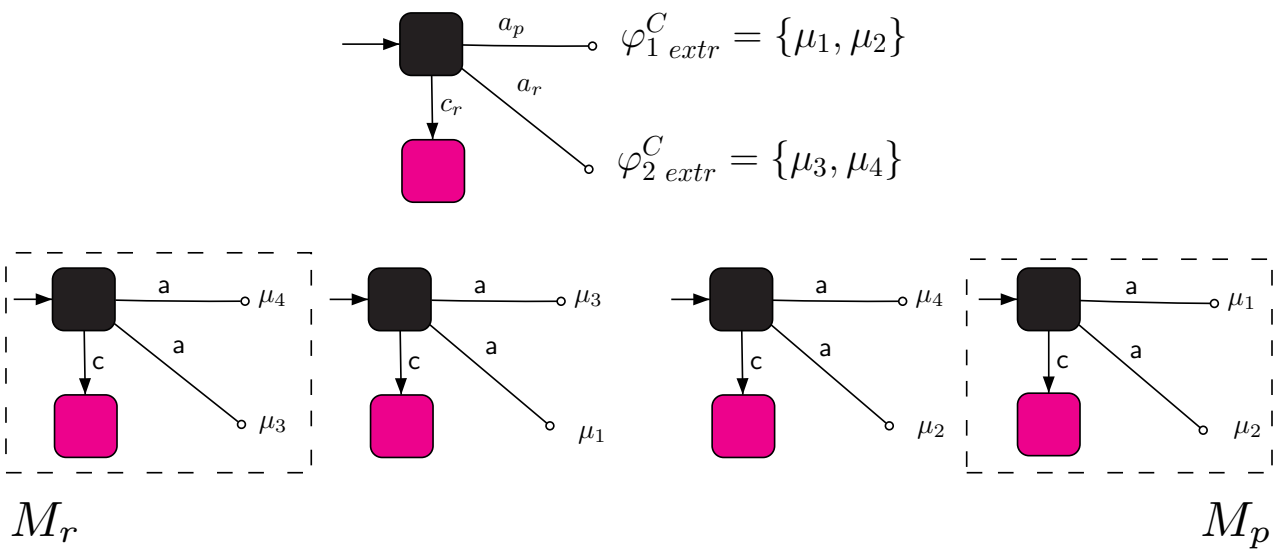

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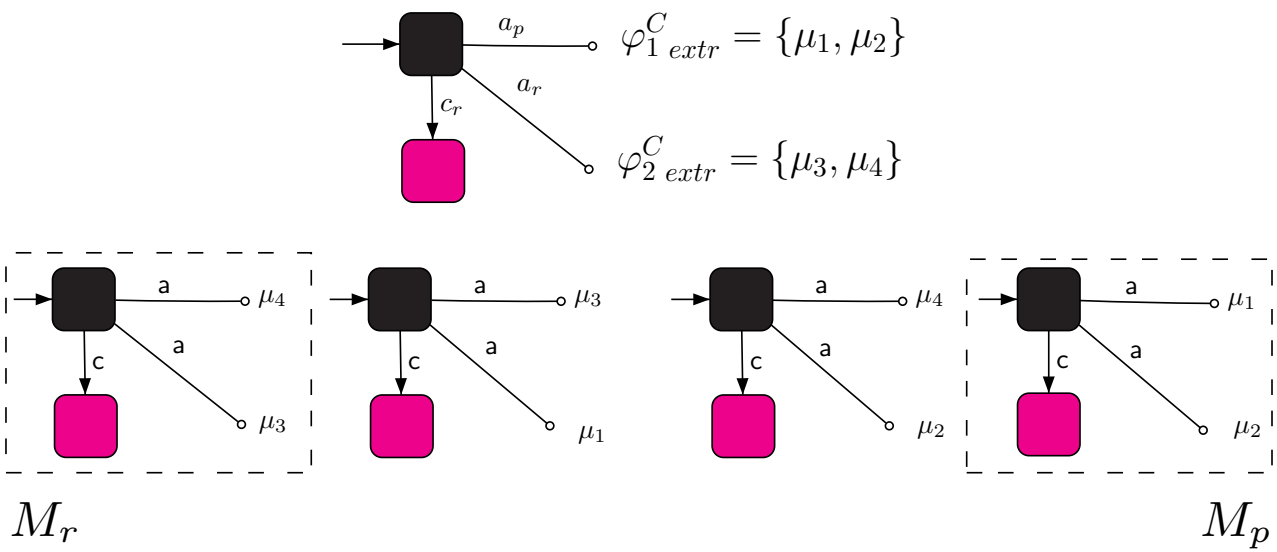

- Analysis of extreme implementations of APA N answers the above question.
- $\mu_1 = [\![.5, .5]\!]$, $\mu_2 = [\![.5, .5]\!]$,
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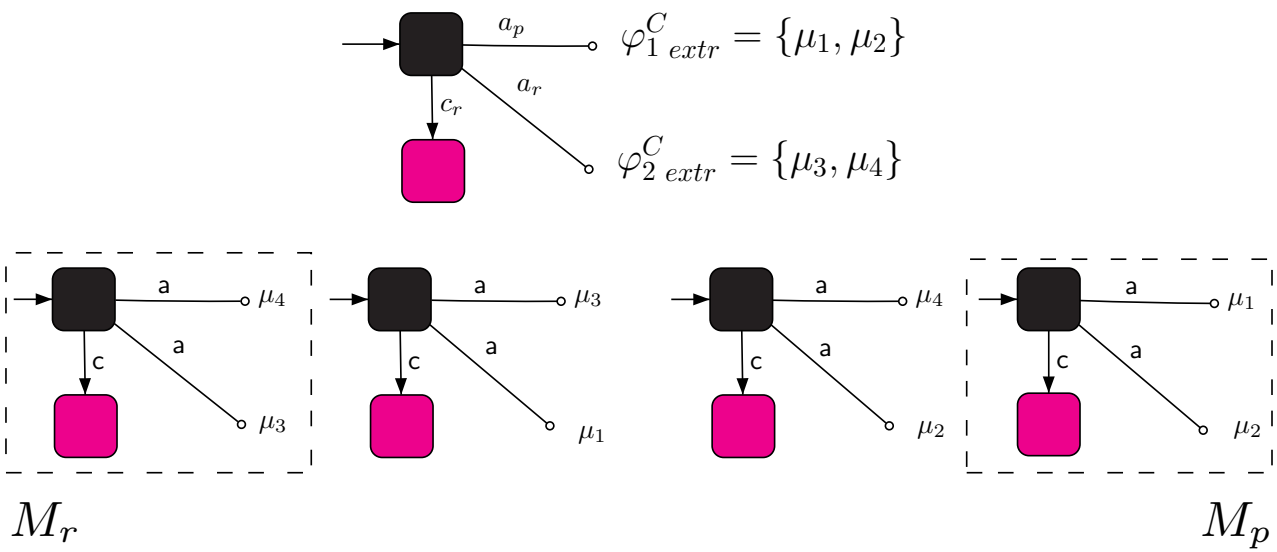

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- **Yes:** $\mu_3(\blacksquare) = .4$ and $\mu_2(\blacksquare) = .5$

Theorems

Theorem

Let M_r and M_p are the extreme implementations of APA N, then for a set G of goal states:

$$Pr_{max}(M_r, G) \leq Pr_{max}(N, G) \leq Pr_{max}(M_p, G)$$

Contents

Abstraction of APA

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Definition

(Multi-transition ($\mapsto$))

$s \overset{a}{\mapsto}_p \varphi'$ iff $s \overset{a}{\rightarrow}_p \varphi_1$, $s \overset{a}{\rightarrow}_p \varphi_2$,..., $s \overset{a}{\rightarrow}_p \varphi_n$ and $\varphi' = \bigvee_{i=1}^{n} \varphi_i$.

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(Abstraction) For APA N, the abstraction function $\alpha : S \to S'$ induces the APA $N' = \alpha(N)$ where for all $s' \in S'$:

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2. $s' \overset{a}{\to}_p \varphi'$ iff $\exists s \in \alpha^{-1}(s')$: $s \overset{a}{\to}_p \varphi$ and $\varphi' = \bigvee_{u \in \alpha^{-1}(s') : u \overset{a}{\to}_p \varphi} \alpha(\varphi)$

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Theorem

$\alpha_1(N_1) \,||_{\bar{A}}\, \alpha_2(N_2) \quad = \quad (\alpha_1 \times \alpha_2)(N_1 ||_{\bar{A}} N_2)$

Refinement of APA

Refinement of APA

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(Refinement ($\preceq$)) For APA N and N', a relation $R \subseteq S \times S'$ is a *refinement* relation if for every sRs':

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$\preceq$ *is a pre-congruence w.r.t. parallel composition.*

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Theorem

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Theorem

For APA N, $N \preceq \alpha(N)$.

Contents

Conclusion

Following are the contributions of my work for APA:

- a novel **compositional abstraction technique** that helps defining **lower/upper** bounds for **maximum** reachability probabilities,
- probabilistic reachability analysis of PA.
- a new notion of a **refinement relation** that is pre-congruence w.r.t parallel composition,

Future work

Future work includes:

- design and implementation of algorithms for abstract-refinement framework for APA.
- lifting the notion of abstraction/refinement from states to distributions.

Thanks