Cinderella versus the wicked Stepmother
or
How to avoid overflow

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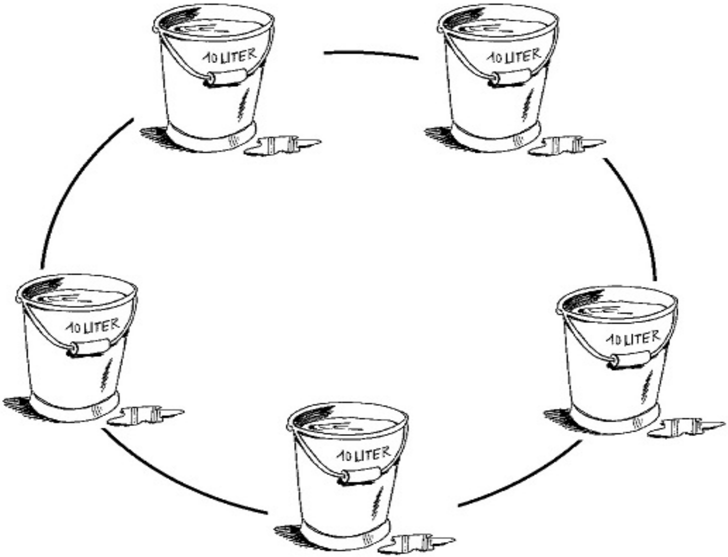
The game



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Question: where is the border, that is, what is the smallest b for which C can avoid overflow?

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  This was a problem in the shortlist of IMO (International Mathematics Olympiad), 2009
- We will generalize the problem to n buckets in a circle, and emptying c consecutive buckets
- Surprise: greedy strategy for C (always empty the two neighboring buckets with the highest amount of water) does *NOT* work for $n = 5$, $c = 2$, as we will show now

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Overflow!

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More general:
Such a lower bound is found by fixing a strategy for S and show that this forces overflow for all (finitely many) choices of C steps

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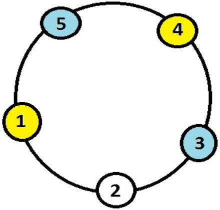
Find *invariant* I such that after every S step executed from a state satisfying I, C can do a step such that again I holds

If $I \Longrightarrow$ all values are $\leq b - 1$, then b is an upper bound

For our case $n = 5$, $c = 2$ after every C step two buckets have been made empty



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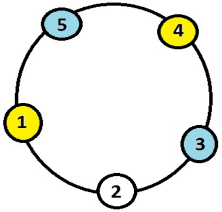

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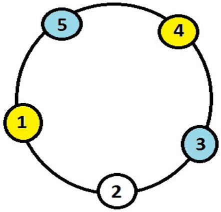

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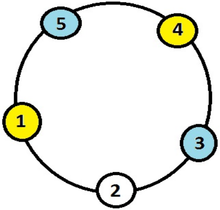

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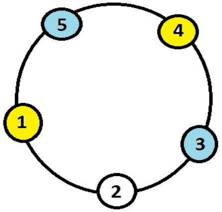

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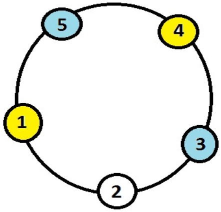

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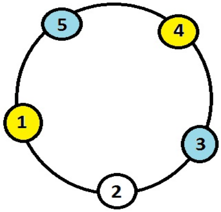

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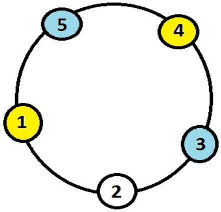

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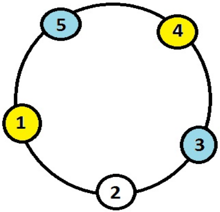

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$I \Rightarrow$ all values are $\leq 2 - 1$, so $b = 2$ is an upper bound

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As C tries to keep the contents of each of the buckets less than $b - 1$, we define $F(n, c) = b - 1$, for $1 \leq c < n$, and b is the lowest value for which C can avoid overflow in the game with n buckets in which at every C step c consecutive buckets are made empty

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So we proved $F(5, 2) = 1$

Results for $F(n, c)$ for $n \leq 12$

	c:	1	2	3	4	5	6	7	8	9	10	11
n:	2	1	–	–	–	–	–	–	–	–	–	–
	3	3/2	1/2	–	–	–	–	–	–	–	–	–
	4	11/6	1	1/3	–	–	–	–	–	–	–	–
	5	25/12	1	5/9	1/4	–	–	–	–	–	–	–
	6	137/60	3/2	1	1/2	1/5	–	–	–	–	–	–
	7	49/20	3/2	1	17/30	7/20	1/6	–	–	–	–	–
	8	363/140	11/6	1	1	1/2	1/3	1/7	–	–	–	–
	9	761/280	11/6	3/2	1	299/525	1/2	9/35	1/8	–	–	–
	10	7129/2520	25/12	3/2	1	1	5/9	69/196	1/4	1/9	–	–
	11	7381/2520	25/12	3/2	1	1	77/135	1/2	1/3	11/54	1/10	–
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Hardest result: $F(n, c) = 1$ for $2 \leq n/c < 3$

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For instance, for proving $F(9,5) \geq 299/525$:

The first phase of the balancing Stepmother ends with four consecutive buckets (say buckets 1, 2, 3, 4) having contents very close to $1/5$. The second phase goes through three further rounds.
In the first of these rounds, the Stepmother uses set $T_1 = \{1,2,3,4,6,7,8\}$ with all buckets except 5 and 9. The Stepmother brings every bucket in T_1 to contents $9/35$. Cinderella leaves a set of four buckets, at least three of which are in T_1. These three buckets are either adjacent (say 2, 3, 4 in this first case) or separated by a single empty bucket (say 3, 4, 6 in the second case).
In the second round the Stepmother selects the set T_2 to contain five buckets; in the first case she uses $T_2 = \{2,3,4,7,8\}$ and in the second case $T_2 = \{3,4,6,7,8\}$. The Stepmother brings every bucket in T_2 to contents $62/175$. Cinderella leaves a set of four buckets, at least two of which are in T_2. We rename the buckets so that 1 and $b \in \{2,3,4\}$ keep their contents $62/175$.

In the third round the Stepmother uses $T_3 = \{1, b, 6\}$, and fills these three buckets up to level $299/525$. Cinderella

must leave at least one such bucket with contents $299/525$ at the end of the round.

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These choices are quite subtle: minor variants of this formula are not invariant any more

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Finally, for all cases with $n \leq 10$ we could give the invariance proofs by hand

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Here the formula I_j is obtained from I by replacing x_i by y_i for $n - c$ consecutive i's, and replacing the other c by 0, for all $j = 1, \ldots, n$ possibilities to do so

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  - find minimal value of b for which C can win
- Definition of game very simple, solutions very hard, already for very low parameters obvious greedy strategy is not optimal

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- Elegant interplay between human proof search and computer support
- It is open whether this approach works for all $0 < c < n$, and whether resulting values are always rational