The Algorithmic Complexity of *k*-domatic Partition of Graphs

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Outline

- Background
  - *k*-domatic partition of a graph
  - applications
  - known complexity results
- Our Contributions
- Conclusions and Open Problems

k-dominating set of a graph

Given a simple undirected graph G =(V,E):

- n:=|V| is the number of vertices in G
- N(v) is the neighborhood of v in G
- d(v):=|N(v)| is the degree of v in G
- $\delta(G):=\min_{v\in V} d(v)$ is the minimum degree of G

S ⊆ V is called a *k*-dominating set of G if:
every vertex in V \ S has at least *k* neighbors in S.

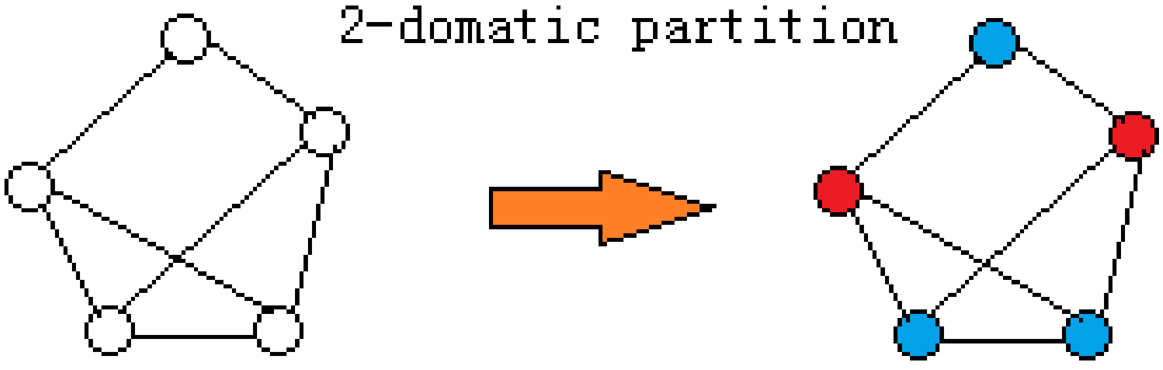
k-dominating set of a graph

- The notion of *k*-dominating set is very useful when higher degree of domination is required.

- For example, in fault-tolerant networks. Suppose we want to locate some servers from which clients can access data. If some servers may fail, then each client should be assigned to more than one servers on which it has access to the data.

k-domatic number of a graph

- A *k*-domatic partition of G is a partition of *V* into disjoint *k*-dominating sets of *G*.
- The *k-domatic number* of *G*, denoted by $d_k(G)$, is the maximum number of *k*-dominating sets in any *k*-domatic partition of *G*.



k-domatic number of a graph

- Domatic number has many applications
  - Distribution of different types of resources
  - Clusterhead rotation in sensor networks
  - ……
- The 1-domatic number has been well studied
  - NP-hard even on bipartite graphs [Garey and Johnson, 1979]
  - Factor $(1+o(1))\ln(n)$ approximation algorithm
  - No $(1-\varepsilon)\ln(n)$ approximation exists unless $NP \subseteq DTIME(n^{O(\log\log(n))})$ [Feige et al., 2002]
- For $k \geq 2$, the algorithmic complexity of k-domatic number has not been well understood.

Our Contributions

Let $k \geq 2$ be a fixed integer. Then:

- It is NP-hard to compute the k-domatic number even on bipartite graphs.
- There is a factor $(1/k + o(1)) \ln(n)$ approximation algorithm for computing the k-domatic number.

  (Recently this ratio is proved to be tight.)
- Determine the exact values of the k-domatic number of special classes of graphs.

NP-hardness of *k*-domatic number

Theorem: Let $k \geq 2$ be a fixed integer. Given a graph G, it is NP-complete to decide whether $d_k(G) \geq 3$.

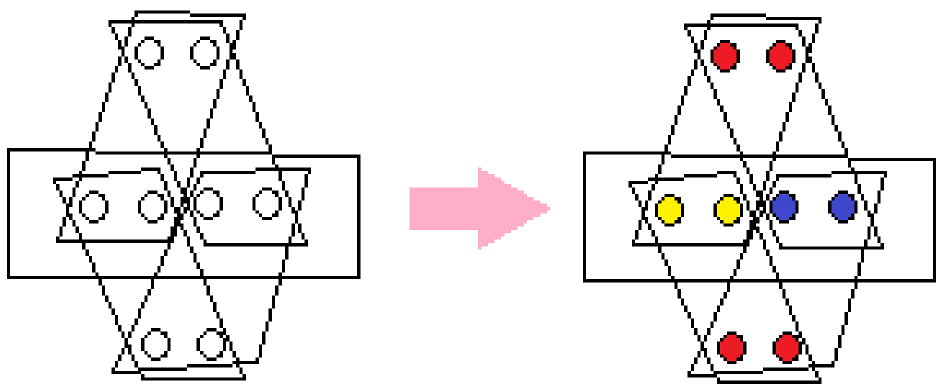
- We first prove the hardness of a hypergraph coloring problem, and then reduce it to the *k*-domatic number problem.
- 2*k*-Uniform Hypergraph Balanced 3-Coloring (2kHB3C):

  Given a 2*k*-uniform hypergraph H, decide whether H has a balanced 3-coloring, i.e., 3-coloring the vertices of H so that each edge contains exactly *k* vertices of one color and *k* vertices of another color.

  (When k=1 it is just the traditional 3-coloring problem.)

NP-hardness of *k*-domatic number

- 2*k*-Uniform Hypergraph Balanced 3-Coloring (2kHB3C):
  Given a 2*k*-uniform hypergraph H, decide whether H has a balanced 3-coloring, i.e., 3-coloring the vertices of H so that each edge contains exactly *k* vertices of one color and *k* vertices of another color.



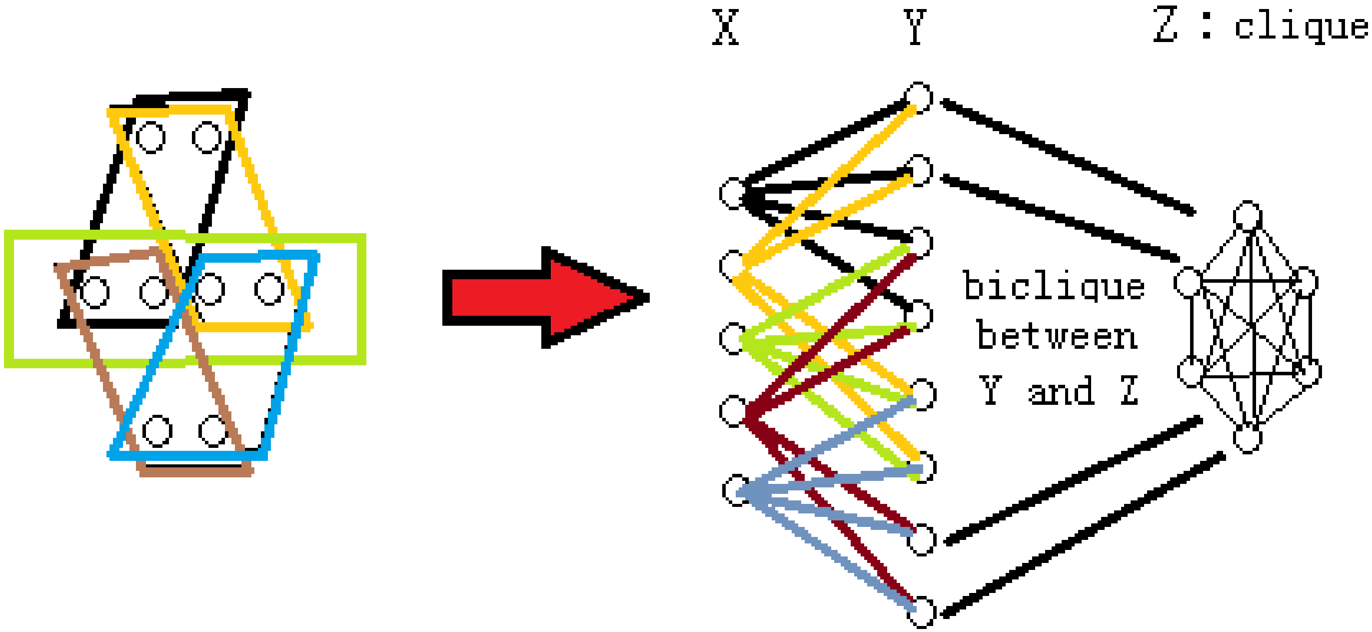
- Proof is omitted here.
- We now reduce 2kHB3C to the problem of deciding whether the *k*-domatic number of a graph is at least 3.

NP-hardness of *k*-domatic number

Let H=(V,E) be an input graph of the 2kHB3C problem.

Construct a graph G = (V', E') as follows:

- $V' = X \cup Y \cup Z$ where $X = \{x_e \mid e \in E\}$, $Y = \{y_v \mid v \in V\}$, $Z = \{z_1, \ldots, z_{3k}\}$.
- $E' = \{\{x_e, y_v\} \mid v \in e \in E\} \cup \{\{y_v, z_i\} \mid v \in V, 1 \le i \le 3k\} \cup \{\{z_i, z_j\} \mid 1 \le i < j \le 3k\}$.
- An example when k=2:



NP-hardness of *k*-domatic number

H has a balanced 3-coloring if and only if $d_k(G) \geq 3$.

"if" direction:

- Suppose (V'_1, V'_2, V'_3) is a k-domatic partition of G.
- Define a coloring c: V(H) → {1,2,3} as: For each $v \in V(H)$, let c(v) = i, where i is the unique integer for which $y_v \in V'_i$.
- Then c is a balanced 3-coloring of H.

- Proof idea: For each edge $e \in E(H)$, assume w.l.o.g. that $x_e \in V'_1$. Since $d(x_e)=2k$, it must have exactly *k* neighbors in V'_2 and *k* in V'_3. Thus, the edge e contains *k* vertices with color 2 and another *k* vertices with color 3.

NP-hardness of *k*-domatic number

H has a balanced 3-coloring if and only if $d_k(G) \geq 3$.

"only if" direction:

- Suppose H has a balanced 3-coloring c: V(H) → {1,2,3}.
- For each edge e ∈ E(H), let R(e) be the color that does not appear in e.
- For i=1,2,3, define:
  $V'_i = \{x_e \mid R(e)=i\} \cup \{y_v \mid c(v)=i\} \cup \{z_j \mid (i-1)k+1 \leq j \leq ik\}$
- Then (V'_1, V'_2, V'_3) is a *k*-domatic partition of G.

Approximation of *k*-domatic number

- Theorem: Given a graph G, we can find in polynomial time a k-domatic partition of G in which the number of k-dominating sets is $(1\text{-}o(1))\ \delta(G)/\ln(n)$.

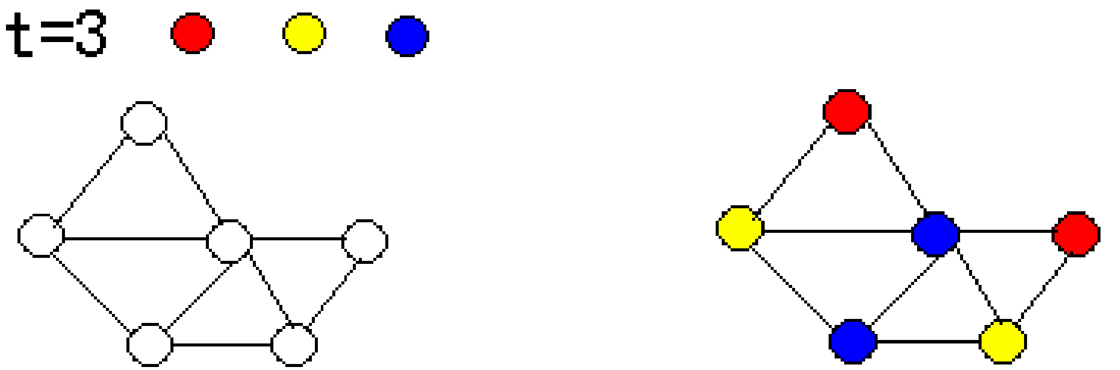
- Because $d_k(G) \leq (\delta(G)/k)+1$, this theorem gives a $(1/k + o(1))\ln(n)$ approximation of $d_k(G)$.

- Proof idea:
  - Randomly partition the graph into many parts, according to some rules.
  - Show that most of the parts in the partition become *k*-dominating sets of G.

Approximation of *k*-domatic number

Let d= δ(G) , and t = d/(ln(n)+3k lnln(n)).

Consider a set of colors {1, 2, …, t}. For each vertex v, color v with one of the t colors chosen uniformly at random.



Partition V into t parts (V_1, V_2, …, V_t), where V_j consists of all the vertices with color j.

Approximation of *k*-domatic number

- We now have a partition $(V_1, V_2, \ldots, V_t)$ of V, where $t = (1-o(1))\, d/\ln(n)$.
- We will prove that the expected number of *k*-dominating sets among these t sets is $(1-o(1))\, t$. If this is true, then we can find a *k*-domatic partition of size $(1-o(1))\, t = (1-o(1))\, d/\ln(n)$, which is precisely what we want.

Approximation of *k*-domatic number

- For each vertex v and color j, let $P(v, j)$ denote the event "v has at most k-1 neighbors of color j."

- If there does NOT exist v for which $P(v, j)$ holds, then each vertex has at least k neighbors in V_j, and thus V_j is a k-dominating set of G.

- Let N be the expected number of pairs (v, j) such that $P(v, j)$ holds. Each such pair prevents at most one set from being k-dominating set. So the expected number of k-dominating sets in the partition is at least $t - N$. Now it suffices to show $N = o(t)$.

Approximation of *k*-domatic number

- If we can show that, for all v and j,

  Pr[P(v, j) holds] ≤ o(1/n),

  then, by linearity of expectation, this implies

  $N = \Sigma_{(v,j)}$ Pr[P(v, j) holds] ≤ (nt) * o(1/n) = o(t).

- P(v, j) means at most k-1 neighbors of v are colored with j. The color of each vertex is given randomly. So we have:

$$\Pr[\mathrm{P}(v, j)\text{ holds}] = \sum_{m=0}^{k-1} \binom{d(v)}{m} \left(\frac{1}{t}\right)^{m} \left(1-\frac{1}{t}\right)^{d(v)-m}$$

  The rest are tedious computations.

Approximation of *k*-domatic number

- Derandomization:

  Method of Conditional Expectations

- Color the vertices one by one.
- For each vertex, try all the t possible colors, and calculate the (conditional) expected number of k-dominating sets in the partition.
  - Important step: the calculation can be done in poly time.
- Choose the color that maximizes the expectation.

Conclusions and Open Problems

- We proved that computing the k-domatic number is NP-hard on bipartite graphs, and give a $(1/k + o(1)) \ln(n)$ approximation algorithm (which is asymptotically tight).
- Open Problems
  - Complexity of computing k-domatic number on special graphs, e.g., interval graphs.
    - 1-domatic number is polynomial on interval graphs. [Lu et al., 1990]
    - It is not known whether k-domatic number, $k \geq 2$, is also polynomial on interval graphs.
  - Other generalizations of domatic number.

Thank you!

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For further discussions, we will give you our instant feedback.