The Uncomputability of the Halting Problem

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Overview

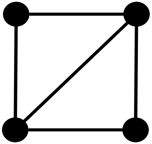
- Decision Problems
- Halting Problem

Decision Problems

- A *decision problem* is a set of instances D and a subset of yes-instances $Y \subseteq D$.
- Usually specified in two parts
  - Generic instance
  - Question

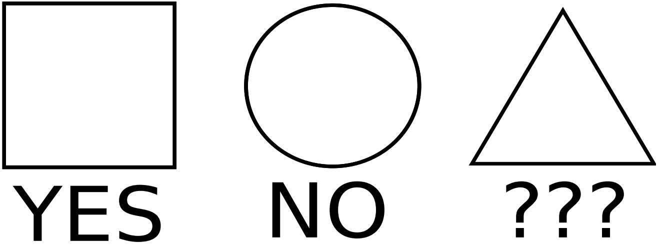
BUY
INSTANCE: Item X
QUESTION: Will I buy X?

PLANARITY
INSTANCE: Graph G
QUESTION: Is G planar?



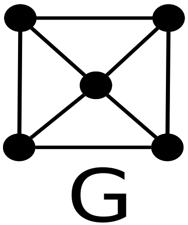
Decision Problems

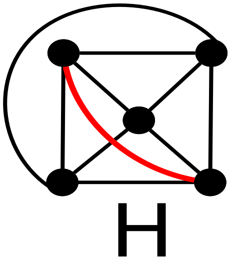
- Some decision problems are computable
- Others may not be



The Halting Problem

HALT
INSTANCE:(α, I)
QUESTION: Does the Turing machine α halt on input I?





The Halting Problem

Theorem

HALT is uncomputable.

Proof Suppose HALT is computable. Then there exists a Turing machine H where

$$H(\alpha, I) = \begin{cases} 1, & \text{if } \alpha \text{ halts on input } I \\ 0, & \textit{otherwise.} \end{cases}$$

So we can construct a Turing machine H' where

$$H'(\alpha) = \begin{cases} 1, & \text{if } H(\alpha, \alpha) = 0 \\ \text{Loops forever}, & \text{if } H(\alpha, \alpha) = 1. \end{cases}$$

The Halting Problem

Consider $H'(H')$.

$$H'(\alpha) = \begin{cases} 1, & \text{if } H(\alpha, \alpha) = 0 \\ \text{Loops forever}, & \text{if } H(\alpha, \alpha) = 1. \end{cases}$$

Case 1

$$\begin{aligned} & H'(H') = 1 \\ & \Rightarrow H(H', H') = 0 \\ & \Rightarrow H' \text{ does not halt on input } H' \\ & \Rightarrow H' \text{ loops forever on input } H', \text{ a contradiction.} \end{aligned}$$

Halting Problem

Consider $H'(H')$

$$H'(\alpha) = \begin{cases} 1, & \text{if } H(\alpha,\alpha) = 0 \\ \text{Loops forever}, & \text{if } H(\alpha,\alpha) = 1. \end{cases}$$

Case 2

$H'(H')$ loops forever
$\Rightarrow H(H', H') = 1$
$\Rightarrow H'$ halts on input H'
$\Rightarrow H'$ does not loop forever, a contradiction.

So the halting problem is not computable.