Weak bisimulations for coalgebras over ordered functors

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Contents

Two equivalent definitions of weak bisimulation for LTS

Let Σ be a set of labels and let $\tau \in \Sigma$ be a silent transition label. Let $\langle A, \Sigma, \to \rangle$ be a labelled transition system.

Definition

A relation $R \subseteq A \times A$ is a *weak bisimulation* if it satisfies the following condition. For $(a, b) \in R$ and $\sigma \neq \tau$ if $a \stackrel{\tau^*}{\to} \circ \stackrel{\sigma}{\to} \circ \stackrel{\tau^*}{\to} a'$ then there is $b' \in A$ such that $b \stackrel{\tau^*}{\to} \circ \stackrel{\sigma}{\to} \circ \stackrel{\tau^*}{\to} b'$ with $(a', b') \in R$ and conversely, for $b \stackrel{\tau^*}{\to} \circ \stackrel{\sigma}{\to} \circ \stackrel{\tau^*}{\to} b''$ there is $a'' \in A$ such that $a \stackrel{\tau^*}{\to} \circ \stackrel{\sigma}{\to} \circ \stackrel{\tau^*}{\to} a''$ and $(a'', b'') \in R$. Moreover, for $\sigma = \tau$ if $a \stackrel{\tau}{\to}^* a'$ then $b \stackrel{\tau}{\to}^* b'$ for some $b' \in B$ with $(a', b') \in R$ and conversely, if $b \stackrel{\tau}{\to}^* b''$ then $a \stackrel{\tau}{\to}^* a''$ for some $a'' \in A$ and $(a'', b'') \in R$.

Two equivalent definitions of weak bisimulation for LTS

Definition

A relation $R \subseteq A \times A$ is a *weak bisimulation* if it satisfies the following condition. If $(a, b) \in R$ then

$$\text{for } \sigma \neq \tau \text{ if } a \xrightarrow{\sigma} a' \text{ then } b \xrightarrow{\tau^*} \circ \xrightarrow{\sigma} \circ \xrightarrow{\tau^*} b' \text{ and } (a', b') \in R,$$

$$\text{for } \sigma = \tau \text{ if } a \xrightarrow{\tau} a' \text{ then } b \xrightarrow{\tau^*} b' \text{ and } (a', b') \in R,$$

$$\text{for } \sigma \neq \tau \text{ if } b \xrightarrow{\sigma} b'' \text{ then } a \xrightarrow{\tau^*} \circ \xrightarrow{\sigma} \circ \xrightarrow{\tau^*} a'' \text{ and } (a'', b'') \in R,$$

$$\text{for } \sigma = \tau \text{ if } b \xrightarrow{\tau} b'' \text{ then } a \xrightarrow{\tau^*} a'' \text{ and } (a'', b'') \in R.$$

Weak bisimulation for different types of transition systems

Weak bisimulation for different transistion systems

For many different types of transistion systems weak bisimulation was defined independently. For instance:

- Labelled transition systems [Milner'80],
- Segala systems [Segala'95],
- and many more...

Contents

Coalgebras as models of systems

Let $F : \mathrm{Set} \to \mathrm{Set}$.

Definition

A coalgebra is a pair $\langle A, \alpha \rangle$, where A is a set and $\alpha : A \to FA$.

Many transition systems are examples of coalgebras:

- deterministic and non-deterministic automata $(F = \{0,1\} \times (-)^{\Sigma}, F = \{0,1\} \times \mathcal{P}((-)^{\Sigma}),$
- Labelled transition systems $(F = \mathcal{P}(\Sigma \times (-)),$
- Segala systems $F = \mathcal{P}(\Sigma \times \mathcal{D}), F = \mathcal{P}(\mathcal{D}(\Sigma \times (-)),$
- and many more...

Bisimulation in coalgebra

Definition

Given two coalgebras $\langle A, \alpha \rangle$ of the same type F, a relation $R \subseteq A \times B$ is a *bisimulation* between the structures if there is a mapping $\gamma : R \to FR$ making the following diagram commutative:

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & B \\ \alpha\downarrow & & \gamma\downarrow & & \downarrow\beta \\ FA & \xleftarrow[F\pi_1]{} & FR & \xrightarrow[F\pi_2]{} & FB \end{array}$$

What about weak bisimulation?

Question

How to introduce the notion of *weak bisimulation* in coalgebra? Is it possible to give two definitions and check their equivalence?

Ordered functors

Let Pos be the category of all posets and monotonic mappings. Note that there is a forgetful functor $U : \mathsf{Pos} \to \mathsf{Set}$ assigning to each poset $(X, \leqslant)$ the underlying set X and to each monotonic map $f : (X, \leqslant) \to (Y, \leqslant)$ the map $f : X \to Y$.

Definition

An ordered functor is a functor $F : \mathsf{Set} \to \mathsf{Pos}$.

This notion is nothing new!

Ordered functors

Let Pos be the category of all posets and monotonic mappings. Note that there is a forgetful functor $U : \mathrm{Pos} \to \mathrm{Set}$ assigning to each poset $(X, \leqslant)$ the underlying set X and to each monotonic map $f : (X, \leqslant) \to (Y, \leqslant)$ the map $f : X \to Y$.

Definition

An ordered functor is a functor $F : \mathrm{Set} \to \mathrm{Pos}$.

This notion is nothing new! To any ordered functor F we assign the composition $\bar{F} = U \circ F$. We identify the ordered functor $F : \mathrm{Set} \to \mathrm{Pos}$ with $\bar{F} = U \circ F : \mathrm{Set} \to \mathrm{Set}$ and write F to denote both F and $\bar{F}$.

Ordered functors

Example

The powerset endofunctor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ can be considered an ordered functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Pos}$ which assigns to any set X the poset $(\mathcal{P}(X), \subseteq)$ and to any map $f : X \to Y$ the order preserving map $\mathcal{P}(f)$.

Ordered functors

Note that if we consider an ordered functor $F : \mathrm{Set} \to \mathrm{Pos}$ then we may introduce for any $X, Y \in \mathrm{Set}$ an order on the set $Hom(X, FY)$ as follows. For $f, g \in Hom(X, FY)$ put

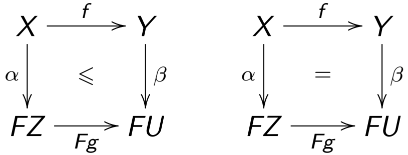
$$f \leqslant g \stackrel{\text{def}}{\iff} f(x) \leqslant_{FY} g(x) \text{ for any } x \in X$$

Ordered functors

Note that if we consider an ordered functor $F : \mathsf{Set} \to \mathsf{Pos}$ then we may introduce for any $X, Y \in \mathsf{Set}$ an order on the set $Hom(X, FY)$ as follows. For $f, g \in Hom(X, FY)$ put

$$f \leqslant g \stackrel{\text{def}}{\iff} f(x) \leqslant_{FY} g(x) \text{ for any } x \in X$$

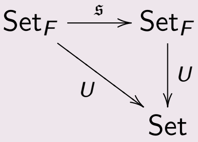
Given $f : X \to Y$, $\alpha : Y \to FZ$, $g : Z \to U$ and $\beta : Y \to FU$ an inequality $Fg \circ \alpha \leqslant \beta \circ f$ will be denoted by a diagram on the left and an equality $Fg \circ \alpha = \beta \circ f$ will be denoted by a diagram on the right:



Coalgebraic operators and coalgebraic saturators

Definition

A *coalgebraic operator* $\mathfrak{s}$ is a functor $\mathfrak{s} : \mathsf{Set}_F \to \mathsf{Set}_F$ such that the following diagram commutes:



Definition

Let $\mathfrak{s} : \mathsf{Set}_F \to \mathsf{Set}_F$ be a coalgebraic operator. We say that $\mathfrak{s}$ is a saturator when it satisfies the following three properties:

- $\alpha \leqslant \mathfrak{s}\alpha$ for any coalgebra $\langle A, \alpha \rangle$ (extensivity),
- $\mathfrak{s} \circ \mathfrak{s} = \mathfrak{s}$ (idempotency),
- if $Ff \circ \alpha \leqslant \beta \circ f$ then $Ff \circ \mathfrak{s}\alpha \leqslant \mathfrak{s}\beta \circ f$ for any $f : X \to Y$ (monotonicity):

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \alpha}\downarrow & \leqslant & \downarrow{\scriptstyle \beta} \\ FA & \xrightarrow[Ff]{} & FB \end{array} \implies \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \mathfrak{s}\alpha}\downarrow & \leqslant & \downarrow{\scriptstyle \mathfrak{s}\beta} \\ FA & \xrightarrow[Ff]{} & FB \end{array}$$

Coalgebraic saturators: examples

Example

Let $\tau \in \Sigma$ be a silent transition label. For a coalgebra structure $\alpha : A \to \mathcal{P}(\Sigma \times A)$ we define its saturation $\mathfrak{s}\alpha : A \to \mathcal{P}(\Sigma \times A)$ as follows. For an element $a \in A$ put

$$\mathfrak{s}\alpha(a) :=$$
$$\alpha(a) \cup \{(\tau, a') \mid a \stackrel{\tau^*}{\to} a'\} \cup \{(\sigma, a') \mid a \stackrel{\tau^*}{\to} \circ \stackrel{\sigma}{\to} \circ \stackrel{\tau^*}{\to} a' \text{ for } \sigma \neq \tau\}$$

$\mathfrak{s}$ is a coalgebraic saturator with respect to the class of all $\mathcal{P}(\Sigma \times (-))$-coalgebras.

Coalgebraic saturators: examples

Example

For a simple Segala system $\langle A, \rightarrow \rangle$ (coalgebra of the type $\mathcal{P}(\Sigma \times \mathcal{D})$), we define a saturated structure $\langle A, \Longrightarrow \rangle$ as follows.Let $\tau \in \Sigma$ be the invisible transition. For any $\sigma \in \Sigma$ we write $a \stackrel{\sigma}{\Longrightarrow}_P \mu$ whenever $\sigma = \tau$ and $\mu \in \mathcal{D}A$ for which $\mu(a) = 1$ or there is a combined step (a, ν) in $\langle A, \alpha \rangle$ such that if $(\sigma', a') \notin \{\sigma, \tau\} \times A$ then $\nu(\sigma', a') = 0$ and $\mu = \sum_{(\sigma', a') \in \{\sigma, \tau\} \times A} \nu(\sigma', a') \cdot \mu_{(\sigma', a')}$ and if $\sigma' = \sigma$ then $a' \stackrel{\tau}{\Longrightarrow}_P \mu_{(\sigma', a')}$ otherwise $\sigma' = \tau$ and $a' \stackrel{\sigma}{\Longrightarrow}_P \mu_{(\sigma', a')}$.

Two approaches to defining weak bisimulation

Definition

A relation $R \subseteq A \times B$ is said to be a *saturated weak bisimulation* between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ provided that there is a structure $\gamma : R \to FR$ for which the following diagram commutes:

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & B \\ \mathfrak{s}\alpha \downarrow & = & \gamma \downarrow & = & \downarrow \mathfrak{s}\beta \\ FA & \xleftarrow[F\pi_1]{} & FR & \xrightarrow[F\pi_2]{} & FB \end{array}$$

Two approaches to defining weak bisimulation

Definition

A relation $R \subseteq A \times B$ is called a *weak bisimilation* provided that there is a structure $\gamma_1 : R \to FR$ and a structure $\gamma_2 : R \to FR$ for which:

- $\alpha \circ \pi_1 = F\pi_1 \circ \gamma_1$ and $F\pi_2 \circ \gamma_1 \leqslant \mathfrak{s}\beta \circ \pi_2$,
- $\beta \circ \pi_2 = F\pi_2 \circ \gamma_2$ and $F\pi_1 \circ \gamma_2 \leqslant \mathfrak{s}\alpha \circ \pi_1$.

$$
\begin{array}{ccccc}
A & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & B \\
\alpha\downarrow & = & \gamma_1\downarrow & \leqslant & \downarrow\mathfrak{s}\beta \\
FA & \xleftarrow[F\pi_1]{} & FR & \xrightarrow[F\pi_2]{} & FB
\end{array}
\qquad
\begin{array}{ccccc}
A & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & B \\
\mathfrak{s}\alpha\downarrow & \geqslant & \gamma_2\downarrow & = & \downarrow\beta \\
FA & \xleftarrow[F\pi_1]{} & FR & \xrightarrow[F\pi_2]{} & FB
\end{array}
$$

Examples

Example

For LTS's definition of a saturated weak bisimulation conicides with the first definition presented at the beginning. The definition of weak bisimulation from previous slide is exactly the 2nd definition of weak bisimulation presented at the beginning of this presentation.

Contents

Weak bisimulation

Theorem

Let $R \subseteq A \times B$ be a standard bisimulation between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$. Then R is a weak bisimilation between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$.

Theorem

If a relation $R \subseteq A \times B$ is a weak bisimulation between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ then $R^{-1} = \{(b, a) \mid (a, b) \in R\}$ is a weak bisimulation between $\langle B, \beta \rangle$ and $\langle A, \alpha \rangle$.

Weak bisimulation

Theorem

If all members of a family $\{R_i\}_{i \in I}$ of relations $R_i \subseteq A \times B$ are weak bisimulations between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ then $\bigcup_{i \in I} R_i$ is also a weak bisimulation between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$.

Theorem

Let $F : \mathsf{Set} \to \mathsf{Set}$ weakly preserve pullbacks and let $\langle A, \alpha \rangle$, $\langle B, \beta \rangle$ and $\langle C, \delta \rangle$ be F-coalgebras from the class C. Let R_1 be a weak bisimulation between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ and R_2 be a weak bisimulation between $\langle B, \beta \rangle$ and $\langle C, \delta \rangle$. Then

$$R_1 \circ R_2 = \{(a, c) \mid \exists b \in B \text{ s.t. } (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$$

is a weak bisimulation between $\langle A, \alpha \rangle$ and $\langle C, \delta \rangle$.

Weak bisimulation

Corollary

If $F : \mathsf{Set} \to \mathsf{Set}$ weakly preserves pullbacks then the greatest weak bisimulation on a coalgebra $\langle A, \alpha \rangle$ is an equivalence relation.

Saturated weak bisimulation

Theorem

Let $R \subseteq A \times B$ be a standard bisimulation between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$. Then R is also a saturated weak bisimilation between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$.

Theorem

A saturated weak bisimulation between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ is defined as a standard bisimulation between saturated models $\langle A, \mathfrak{s}\alpha \rangle$ and $\langle B, \mathfrak{s}\beta \rangle$. Hence, any property true for standard bisimulation is also true for a saturated weak bisimulation.

Weak and saturated weak bisimulation

Theorem

Let $F : \mathsf{Set} \to \mathsf{Set}$ weakly preserve kernel pairs and let $R \subseteq A \times A$ be an equivalence relation which is a weak bisimulation on $\langle A, \alpha \rangle$. Then R is a saturated weak bisimulation on $\langle A, \alpha \rangle$.

We say that two elements $a, b \in A$ are weakly bisimilar, and write $a \approx_w b$ if there is a weak bisimulation $R \subseteq A \times A$ on $\langle A, \alpha \rangle$ for which $(a, b) \in R$. We say that a and b are saturated weakly bisimilar, and write $a \approx_{sw} b$, if there is a saturated weak bisimulation R on $\langle A, \alpha \rangle$ containing (a, b).

Corollary

Let $F : \mathsf{Set} \to \mathsf{Set}$ be a functor weakly preserving pullbacks. Then the relations $\approx_w$ and $\approx_{sw}$ are equivalence relations and

$$\approx_w \subseteq \approx_{sw} .$$

Saturated weak and weak bisimulation

Definition

We say that an ordered functor $F : \mathrm{Set} \to \mathrm{Pos}$ *preserves downsets* provided that for any $f : X \to Y$ and any $\vec{x} \in FX$ the following equality holds:

$$Ff(\vec{x}\downarrow) = Ff(\{\vec{x}' \in FX \mid \vec{x}' \leqslant \vec{x}\}) = Ff(\vec{x})\downarrow = \{\vec{y} \in FY \mid \vec{y} \leqslant Ff(\vec{x})\}.$$

Example

The powerset functor $\mathcal{P}$ preserves downsets.

Saturated weak and weak bisimulation

Theorem

Let $F : \text{Set} \to \text{Set}$ *weakly preserve kernel pairs and preserve downsets. Let* $R \subseteq A \times A$ *be an equivalence relation which is a saturated weak bisimulation on* $\langle A, \alpha \rangle$*. Then* R *is a weak bisimulation on* $\langle A, \alpha \rangle$*.*

Weak and saturated weak bisimulation

Corollary

Let $F : \mathsf{Set} \to \mathsf{Set}$ *weakly preserve pullbacks and preserve downsets. Then for any* F*-coalgebra* $\langle A, \alpha \rangle \in \mathsf{C}$ *the relations* $\approx_w$ *and* $\approx_{sw}$ *are equivalence relations and*

$$\approx_w = \approx_{sw} .$$

Contents

Terminal object in Set_F^w

Let Set_F^w denote the category of all F-coalgebras as objects and all maps $f : A \to B$ between $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ for which the relation $\{(a, f(a) \mid a \in A\}$ is a weak bisimulation between $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ as morphisms.

Theorem

Let F weakly preserve pullbacks. If $\langle T, \tau\rangle$ is the terminal object in Set_F then the greatest subcoalgebra $\langle T_s, \tau_s\rangle$ of $\langle T, \tau\rangle$ closed under saturation, i.e. $\mathfrak{s}\tau_s = \tau_s$ is the terminal object in Set_F^w.

Weak coinduction principle

Weak coinduction principle

Let F weakly preserve pullbacks. Let $\langle A, \alpha \rangle$ be any F-coalgebra and let $[\![-]\!]^w_\alpha$ denote the unique weak homomorphism from $\langle A, \alpha \rangle$ to $\langle T_s, \tau_s \rangle$. For two elements $a, b \in A$ we have

$$a \approx_w b \iff [\![a]\!]^w_\alpha = [\![b]\!]^w_\alpha$$

Contents

Problem with saturators...

The saturator defined in this talk is more general than expected.

Example

Simplest saturator is $\mathcal{I}d : \mathsf{Set}_F \to \mathsf{Set}_F$. Here, saturated weak bisimulation is a standard bisimulation.

Example

For an LTS $\langle A, \alpha \rangle$ put

$$\mathfrak{s}\alpha(a) = \{(\tau, a)\} \cup \{(\sigma, a') \mid a \overset{\tau^*}{\Longrightarrow} \circ \overset{\sigma}{\to} a'\}$$

Here, $\mathfrak{s}$ is associated with delay bisimulation.

The saturation algebra

For any $\alpha, \alpha' : A \to FA$ define the following operations:

$$\alpha + \alpha' = \alpha \vee \alpha',$$
$$\alpha \triangleright \alpha',$$
$$\alpha \triangleleft \alpha',$$
$$0 := \perp,$$
$$1_A$$
$$\alpha^* := \min\{\beta \geqslant \alpha \mid \beta = 1 + \beta + \beta \triangleleft \beta + \beta \triangleright \beta\}(\text{weak bis. sat.}),$$
$$\alpha^d :=:= \min\{\beta \geqslant \alpha \mid \beta = 1 + \beta + \beta \triangleleft \beta\}(\text{delay bis. sat.}).$$

We get a saturation algebra

$$(Hom(A, FA), +, \triangleleft, \triangleright, (-)^*, (-)^d, 0, 1).$$

Thank you for your attention!

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